

Martin Eichler
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**The Theory of
Jacobi Forms**

1985

Birkhäuser
Boston · Basel · Stuttgart

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Library of Congress Cataloging in Publication Data

Eichler, M. (Martin)
The theory of Jacobi forms.

(Progress in mathematics ; v. 55)

Bibliography: p.

1. Jacobi forms. 2. Forms, Modular. I. Zagier,
Don, 1951-. II. Title. III. Series: Progress in
mathematics (Boston, Mass.) ; v. 55.

QA243.E36 1985 512.9'44 84-28250
ISBN 0-8176-3180-1

CIP-Kurztitelaufnahme der Deutschen Bibliothek

Eichler, Martin:

The theory of Jacobi forms / Martin Eichler ; Don Zagier. -
Boston ; Basel ; Stuttgart : Birkhäuser, 1985.

(Progress in mathematics ; Vol. 55)

ISBN 3-7643-3180-1 (Basel ...)

ISBN 0-8176-3180-1 (Boston)

NE: Zagier, Don B.; GT

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Printed in Germany

ISBN 0-8176-3180-1

ISBN 3-7643-3180-1

9 8 7 6 5 4 3 2 1

T A B L E O F C O N T E N T S

Introduction	1
Notations	7
I. Basic Properties	8
1. Jacobi forms and the Jacobi group	8
2. Eisenstein series and cusp forms	17
3. Taylor expansions of Jacobi forms	28
Application: Jacobi forms of index one	37
4. Hecke operators	41
II. Relations with Other Types of Modular Forms	57
5. Jacobi forms and modular forms of half-integral weight	57
6. Fourier-Jacobi expansions of Siegel modular forms and the Saito-Kurokawa conjecture	72
7. Jacobi theta series and a theorem of Waldspurger .	81
III. The Ring of Jacobi Forms	89
8. Basic structure theorems	89
9. Explicit description of the space of Jacobi forms.	100
Examples of Jacobi forms of index greater than 1 .	113
10. Discussion of the formula for $\dim J_{k,m}$	121
11. Zeros of Jacobi forms	133
Tables	141
Bibliography	146

INTRODUCTION

The functions studied in this monograph are a cross between elliptic functions and modular forms in one variable. Specifically, we define a *Jacobi form on $SL_2(\mathbb{Z})$* to be a holomorphic function

$$\phi: \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C} \quad (\mathcal{H} = \text{upper half-plane})$$

satisfying the two transformation equations

$$(1) \quad \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i mcz}{c\tau+d}} \phi(\tau, z) \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\right)$$

$$(2) \quad \phi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \phi(\tau, z) \quad (\lambda, \mu) \in \mathbb{Z}^2$$

and having a Fourier expansion of the form

$$(3) \quad \phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm}} c(n, r) e^{2\pi i (n\tau + rz)} .$$

Here k and m are natural numbers, called the *weight* and *index* of ϕ , respectively. Note that the function $\phi(\tau, 0)$ is an ordinary modular form of weight k , while for fixed τ the function $z \mapsto \phi(\tau, z)$ is a function of the type normally used to embed the elliptic curve $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ into a projective space.

If $m = 0$, then ϕ is independent of z and the definition reduces to the usual notion of modular forms in one variable. We give three other examples of situations where functions satisfying (1)–(3) arise classically:

1. Theta series. Let $Q: \mathbb{Z}^N \rightarrow \mathbb{Z}$ be a positive definite integer valued quadratic form and B the associated bilinear form. Then for any vector $x_0 \in \mathbb{Z}^N$ the theta series

$$(4) \quad \theta_{x_0}(\tau, z) = \sum_{x \in \mathbb{Z}^N} e^{2\pi i (Q(x)\tau + B(x, x_0)z)}$$

is a Jacobi form (in general on a congruence subgroup of $\text{SL}_2(\mathbb{Z})$) of weight $N/2$ and index $Q(x_0)$; the condition $r^2 \leq 4nm$ in (3) arises from the fact that the restriction of Q to $\mathbb{Z}x + \mathbb{Z}x_0$ is a positive definite binary quadratic form. Such theta series (for $N=1$) were first studied by Jacobi [10], whence our general name for functions satisfying (1) and (2).

2. Fourier coefficients of Siegel modular forms. Let $F(Z)$ be a Siegel modular form of weight k and degree 2. Then we can write Z as $\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $z \in \mathbb{C}$, $\tau, \tau' \in \mathcal{H}$ (and $\text{Im}(z)^2 < \text{Im}(\tau)\text{Im}(\tau')$), and the function F is periodic in each variable τ , z and τ' . Write its Fourier expansion with respect to τ' as

$$(5) \quad F(Z) = \sum_{m=0}^{\infty} \phi_m(\tau, z) e^{2\pi i m \tau'} ;$$

then for each m the function ϕ_m is a Jacobi form of weight k and index m , the condition $4nm \geq r^2$ in (3) now coming from the fact that F has a Fourier development of the form $\sum c(T) e^{2\pi i \text{Tr}(TZ)}$ where T ranges over positive semi-definite symmetric 2×2 matrices. The expansion (5) (and generalizations to other groups) was first studied by Piatetski-Shapiro [26], who referred to it as the *Fourier-Jacobi expansion* of F and to the coefficients ϕ_m as Jacobi functions, a word which we will reserve for (meromorphic) quotients of Jacobi forms of the same weight and index, in accordance with the usual terminology for modular forms and functions.

3. The Weierstrass p -function. The function

$$(6) \quad p(\tau, z) = z^{-2} + \sum_{\substack{\omega \in \mathbb{Z} + \mathbb{Z}\tau \\ \omega \neq 0}} ((z + \omega)^{-2} - \omega^{-2})$$

is a meromorphic Jacobi form of weight 2 and index 0; we will see

later how to express it as a quotient of holomorphic Jacobi forms (of index 1 and weights 12 and 10).

Despite the importance of these examples, however, no systematic theory of Jacobi forms along the lines of Hecke's theory of modular forms seems to have been attempted previously.* The authors' interest in constructing such a theory arose from their attempts to understand and extend Maass' beautiful work on the "Saito-Kurokawa conjecture". This conjecture, formulated independently by Saito and by Kurokawa [15] on the basis of numerical calculations of eigenvalues of Hecke operators for the (full) Siegel modular group, asserted the existence of a "lifting" from ordinary modular forms of weight $2k-2$ (and level one) to Siegel modular forms of weight k (and also level one); in a more precise version, it said that this lifting should land in a specific subspace of the space of Siegel modular forms (the so-called Maass "Spezialschar", defined by certain identities among Fourier coefficients) and should in fact be an isomorphism from $M_{2k-2}(SL_2(\mathbb{Z}))$ onto this space, mapping Eisenstein series to Eisenstein series, cusp forms to cusp forms, and Hecke eigenforms to Hecke eigenforms. Most of this conjecture was proved by Maass [21,22,23], another part by Andrianov [2], and the remaining part by one of the authors [40]. It turns out that the

* Shimura [31,32] has studied the same functions and also their higher-dimensional generalizations. By multiplication by appropriate elementary factors they become modular functions in τ and elliptic (resp. Abelian) functions in z , although non-analytic ones. Shimura used them for a new foundation of complex multiplication of Abelian functions. Because of the different aims Shimura's work does not overlap with ours. We also mention the work of R.Berndt [3,4], who studied the quotient field (field of Jacobi functions) from both an algebraic-geometrical and arithmetical point of view. Here, too, the overlap is slight since the field of Jacobi functions for $SL_2(\mathbb{Z})$ is easily determined (it is generated over \mathbb{C} by the modular invariant $j(\tau)$ and the Weierstrass p -function $p(\tau, z)$); Berndt's papers concern Jacobi functions of higher level. Finally, the very recent paper of Feingold and Frenkel [Math. Ann. 263, 1983] on Kac-Moody algebras uses functions equivalent to our Jacobi forms, though with a very different motivation; here there is some overlap of their results and our §9 (in particular, our Theorem 9.3 seems to be equivalent to their Corollary 7.11).

conjectured correspondence is the composition of three isomorphisms

$$\begin{array}{c}
 \text{Maass "Spezialschar" } \subset M_k(Sp_4(\mathbb{Z})) \\
 \downarrow \\
 \text{Jacobi forms of weight } k \text{ and index } 1 \\
 (7) \quad \downarrow \\
 \text{Kohnen's "+"-space ([11]) } \subset M_{k-\frac{1}{2}}(\Gamma_0(4)) \\
 \downarrow \\
 M_{2k-2}(SL_2(\mathbb{Z})) \quad ;
 \end{array}$$

the first map associates to each F the function ϕ_1 defined by (5), the second is given by

$$\sum_{n \geq 0} c(n) e^{2\pi i n \tau} \rightarrow \sum_{n \geq 0} \sum_{r^2 \leq 4n} c(4n - r^2) e^{2\pi i (n\tau + rz)} ,$$

and the third is the Shimura correspondence [29,30] between modular forms of integral and half-integral weight, as sharpened by Kohnen [11] for the case of forms of level 1.

One of the main purposes of this work will be to explain diagram (7) in more detail and to discuss the extent to which it generalizes to Jacobi forms of higher index. This will be carried out in Chapters I and II, in which other basic elements of the theory (Eisenstein series, Hecke operators, ...) are also developed. In Chapter III we will study the bigraded ring of all Jacobi forms on $SL_2(\mathbb{Z})$. This is much more complicated than the usual situation because, in contrast with the classical isomorphism $M_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$, the ring $J_{*,*} = \bigoplus_{k,m} J_{k,m}$ ($J_{k,m}$ = Jacobi forms of weight k and index m) is not finitely generated. Nevertheless, we will be able to obtain considerable information about the structure of $J_{*,*}$. In particular, we will find upper and lower bounds for $\dim J_{k,m}$ which agree for k sufficiently large ($k \geq m$), will prove that $J_{*,m} = \bigoplus_k J_{k,m}$ is a free module of rank $2m$ over the ring $M_*(SL_2(\mathbb{Z}))$, and will describe explicit algorithms for finding

bases of $J_{k,m}$ as a vector space over \mathbb{C} and of $J_{*,m}$ as a module over $M_*(SL_2(\mathbb{Z}))$. The dimension formula obtained has the form

$$(8) \quad \dim J_{k,m} = \sum_{r=0}^m \dim M_{k+2r} - N(m)$$

for k even (and sufficiently large), where $N(m)$ is given by

$$N(m) = \sum_{r=0}^m \left\lceil \frac{r^2}{4m} \right\rceil \quad (\lceil x \rceil = \text{smallest integer } \geq x).$$

We will show that $N(m)$ can be expressed in terms of class numbers of imaginary quadratic fields and that (8) is equivalent to the formula

$$(9) \quad \dim J_{k,m}^{\text{new}} = \dim M_{2k-2}^{\text{new}}(\Gamma_0(m))^+,$$

where $M_{2k-2}^{\text{new}}(\Gamma_0(m))^+$ is the space of new forms of weight $2k-2$ on $\Gamma_0(m)$ which are invariant under the Atkin-Lehner (or Fricke) involution $f(\tau) \mapsto m^{-k+1} \tau^{-2k+2} f(-1/m\tau)$ and $J_{k,m}^{\text{new}}$ a suitably defined space of "new" Jacobi forms.

Chapter IV, which will be published as a separate work, goes more deeply into the Hecke theory of Jacobi forms. In particular, it is shown with the aid of a trace formula that the equality of dimensions (9) actually comes from an isomorphism of the corresponding spaces as modules over the ring of Hecke operators.

Another topic which will be treated in a later paper (by B.Gross, W.Kohnen and the second author) is the relationship of Jacobi forms to Heegner points. These are specific points on the modular curve $X_0(m) = \mathcal{H}/\Gamma_0(m) \cup \{\text{cusps}\}$ (namely, those satisfying a quadratic equation with leading coefficient divisible by m). It turns out that for each n and r with $r^2 < 4nm$ one can define in a natural way a class $P(n,r) \in \text{Jac}(X_0(m))(\mathbb{Q})$ as a combination of Heegner points and cusps and

that the sum $\sum_{n,r} P(n,r) q^n \zeta^r$ is an element of $\text{Jac}(X_0(m))(\mathbb{Q}) \otimes_{\mathbb{Q}} J_{2,m}$.

One final remark. Since this is the first work on the theory of Jacobi forms, we have tried to give as elementary and understandable an exposition as possible. This means in particular that we have always preferred a more classical to a more modern approach (for instance, Jacobi forms are defined by transformation equations in $\mathcal{H} \times \mathbb{C}$ rather than as sections of line bundles over a surface or in terms of the representation theory of Weil's metaplectic group), that we have often given two proofs of the same result if the shorter one seemed to be too uninformative or to depend too heavily on special properties of the full modular group, and that we have included a good many numerical examples. Presumably the theory will be developed at a later time from a more sophisticated point of view.

* * * *

This work originated from a much shorter paper by the first author submitted for publication early in 1980. In this the Saito-Kurokawa conjecture was proved for modular (Siegel and elliptic) forms on $\Gamma_0(N)$ with arbitrary level N. However, the exact level of the forms in the bottom of diagram (7) was left open. The procedure was about the same as here in §§4-6. The second author persuaded the first to withdraw his paper and undertake a joint study in a much broader frame. Sections 2 and 8-10 are principally due to the second author, while sections 1, 3-7 and 11 are joint work.

The authors would like to thank G. van der Geer for his critical reading of the manuscript.

Notations

We use \mathbb{N} to denote the set of natural numbers, \mathbb{N}_0 for $\mathbb{N} \cup \{0\}$. We use Knuth's notation $[x]$ (rather than the usual $\lfloor x \rfloor$) for the greatest-integer function $\max\{n \in \mathbb{Z} \mid n \leq x\}$ and similarly $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\} = -\lfloor -x \rfloor$. The symbol \square denotes any square number. By $d \parallel n$ we mean $d \mid n$ and $(d, \frac{n}{d}) = 1$. In sums of the form $\sum_{\substack{d \mid n \\ d \neq \lambda}}$ it is understood that the summation is over positive divisors only. The function $\sum_{d \mid n} d^y$ ($d \in \mathbb{N}$) is denoted $\sigma_y(n)$.

The symbol $e(x)$ denotes $e^{2\pi i x}$, while $e^m(x)$ and $e_m(x)$ ($m \in \mathbb{N}$) denote $e(mx)$ and $e(x/m)$, respectively. In $e(x)$ and $e^m(x)$, x is a complex variable, but in $e_m(x)$ it is to be taken in $\mathbb{Z}/m\mathbb{Z}$; thus $e_m(ab^{-1})$ means $e_m(n)$ with $bn \equiv a \pmod{m}$, and not $e(a/bm)$.

We use M^t and I_n for the transpose of a matrix and for the $n \times n$ identity matrix, respectively. The symbol $[a, b, c]$ denotes the quadratic form $ax^2 + bxy + cy^2$.

\mathcal{H} denotes the upper half-plane $\{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$. The letters τ and z will always be reserved for variables in \mathcal{H} and \mathbb{C} , respectively, with $\tau = u + iv$, $z = x + iy$, $q = e(\tau)$, $\zeta = e(z)$. The group $SL_2(\mathbb{Z})$ will often be denoted by Γ_1 and the space of modular (resp. cusp) forms of weight k on Γ_1 by M_k (resp. S_k). The normalized Eisenstein series $E_k \in M_k$ ($k \geq 4$ even) are defined in the usual way; in particular one has $M_* := \bigoplus_k M_k = \mathbb{C}[E_4, E_6]$ with $E_4 = 1 + 240 \sum \sigma_3(n)q^n$, $E_6 = 1 - 504 \sum \sigma_5(n)q^n$.

The symbol " $:=$ " means that the expression on the right is the definition of that on the left.

Chapter I
BASIC PROPERTIES

§1. Jacobi Forms and the Jacobi Group

The definition of Jacobi forms for the full modular group $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ was already given in the Introduction. In order to treat subgroups $\Gamma \subset \Gamma_1$ with more than one cusp, we have to rewrite the definition in terms of an action of the groups $\mathrm{SL}_2(\mathbb{Z})$ and \mathbb{Z}^2 on functions $\phi: \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C}$. This action, analogous to the action

$$(1) \quad (f|_k^M)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) \quad (M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1)$$

in the usual theory of modular forms, will be important for several later constructions (Eisenstein series, Hecke operators). We fix integers k and m and define

$$(2) \quad \left(\phi|_{k,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)(\tau, z) := (c\tau + d)^{-k} e^m\left(\frac{-cz^2}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1\right)$$

and

$$(3) \quad (\phi|_m[\lambda \ \mu])(\tau, z) := e^{m(\lambda^2\tau + 2\lambda z)} \phi(\tau, z + \lambda\tau + \mu) \quad ((\lambda \ \mu) \in \mathbb{Z}^2),$$

where $e^m(x) = e^{2\pi i mx}$ (see "Notations"). Thus the two basic transformation laws of Jacobi forms can be written

$$\phi|_{k,m}^M = \phi \quad (M \in \Gamma_1), \quad \phi|_m^X = \phi \quad (X \in \mathbb{Z}^2),$$

where we have dropped the square brackets around M or X to lighten the notation. One easily checks the relations

$$(4) \quad \begin{aligned} (\phi|_{k,m} M)|_{k,m} M' &= \phi|_{k,m} (MM') , & (\phi|_m X)|_m X' &= \phi|_m (X+X') , \\ (\phi|_{k,m} M)|_m XM &= (\phi|_m X)|_{k,m} M , & (M, M' \in \Gamma_1, X, X' \in \mathbb{Z}^2) . \end{aligned}$$

They show that (2) and (3) jointly define an action of the semi-direct product $\Gamma_1^J := \Gamma_1 \ltimes \mathbb{Z}^2$ (= set of products (M, X) with $M \in \Gamma_1$, $X \in \mathbb{Z}^2$ and group law $(M, X)(M', X') = (MM', XM' + X')$; notice that we are writing our vectors as row vectors, so Γ_1 acts on the right), the (full) Jacobi group. We will discuss this action in more detail at the end of this section.

We can now give the general definition of Jacobi forms.

Definition. A Jacobi form of weight k and index m ($k, m \in \mathbb{N}$) on a subgroup $\Gamma \subset \Gamma_1$ of finite index is a holomorphic function $\phi: \mathcal{H} \times \mathbb{C} \rightarrow$ satisfying

- i) $\phi|_{k,m} M = \phi$ ($M \in \Gamma$);
- ii) $\phi|_m X = \phi$ ($X \in \mathbb{Z}^2$);
- iii) for each $M \in \Gamma_1$, $\phi|_{k,m} M$ has a Fourier development of the form $\sum c(n, r) q^n \zeta^r$ ($q = e(\tau)$, $\zeta = e(z)$) with $c(n, r) = 0$ unless $n \geq r^2/4m$. (If ϕ satisfies the stronger condition $c(n, r) \neq 0 \Rightarrow n > r^2/4m$, it is called a cusp form.)

The vector space of all such functions ϕ is denoted $J_{k,m}(\Gamma)$; if $\Gamma = \Gamma_1$ we write simply $J_{k,m}$ for $J_{k,m}(\Gamma_1)$.

Remarks. 1. The numbers n, r in iii) are in general in \mathbb{Q} , not in \mathbb{Z} (but with bounded denominator, depending on Γ and M).

2. We could define Jacobi forms with character, $J_{k,m}(\Gamma, X)$, by inserting a factor $X(M)$ in i) in the usual way.

3. Also, we could replace \mathbb{Z}^2 by any lattice invariant under Γ , e.g. by imposing congruence conditions modulo N if $\Gamma = \Gamma(N)$. It would therefore be more proper to refer to functions satisfying i)-iii)

as Jacobi forms on the Jacobi group $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$ (rather than on Γ).

However, we will not worry about this since most of the time we will be concerned only with the full Jacobi group.

Our first main result is

THEOREM 1.1. *The space $J_{k,m}(\Gamma)$ is finite-dimensional.*

This will follow from two other results, both of independent interest:

THEOREM 1.2. *Let ϕ be a Jacobi form of index m . Then for fixed $\tau \in \mathcal{H}$, the function $z \mapsto \phi(\tau, z)$, if not identically zero, has exactly $2m$ zeros (counting multiplicity) in any fundamental domain for the action of the lattice $\mathbb{Z}\tau + \mathbb{Z}$ on \mathbb{C} .*

Proof. It follows easily from the transformation law ii) that

$$\frac{1}{2\pi i} \oint_{\partial F} \frac{\phi_z(\tau, z)}{\phi(\tau, z)} dz = 2m \quad (\phi_z = \frac{\partial \phi}{\partial z}, \quad F = \text{fundamental domain for } \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z})$$

(the expression $\frac{1}{2\pi i} \frac{\phi_z}{\phi}$ is invariant under $z \rightarrow z+1$ and changes by $2m$ when one replaces z by $z+\tau$), and this is equivalent to the statement of the theorem. Notice that the same proof works for ϕ meromorphic (with "number of zeros" replaced by "number of zeros minus number of poles") and any $m \in \mathbb{Z}$. A consequence is that there are no holomorphic Jacobi forms of negative index, and that a holomorphic Jacobi form of index 0 is independent of z (and hence simply an ordinary modular form of weight k in τ).

THEOREM 1.3. *Let ϕ be a Jacobi form on Γ of weight k and index m and λ, μ rational numbers. Then the function $f(\tau) = e^{m(\lambda^2\tau)} \phi(\tau, \lambda\tau + \mu)$ is a modular form (of weight k and on some subgroup of Γ' of finite index depending only on Γ and on λ, μ).*

For $\lambda = \mu = 0$ it is clear that $\tau \mapsto \phi(\tau, 0)$ is a modular form of weight k on Γ . We will prove the general case later on in this section when we have developed the formalism of the action of the Jacobi group further. Note that the Fourier development of $f(\tau)$ at infinity is

$$\sum_{n,r} e(r\mu)c(n,r) e((m\lambda^2 + r\lambda + n)\tau) ,$$

so that the conditions $n \geq 0$, $r^2 \leq 4nm$ in the definition of Jacobi forms are exactly what is required to ensure the holomorphicity of f at ∞ in the usual sense.

To deduce 1.1, we pick any $2m$ pairs of rational numbers $(\lambda_i, \mu_i) \in \mathbb{Q}^2$ with $(\lambda_i, \mu_i) \not\equiv (\lambda_j, \mu_j) \pmod{\mathbb{Z}^2}$ for $i \neq j$. Then the functions $f_i(\tau) = e^{m(\lambda_i^2\tau)}\phi(\tau, \lambda_i\tau + \mu_i)$ lie in $M_k(\Gamma_i)$ for some subgroups Γ_i of Γ , and the map $\phi \mapsto \{f_i\}_i$ is injective by Theorem 1.2. Therefore $\dim J_{k,m}(\Gamma) \leq \sum_i \dim M_k(\Gamma_i)$; this proves Theorem 1.1 and also shows that $J_{k,m}$ is 0 for $k \leq 0$ unless $k = m = 0$, in which case it reduces to the constants.

To prove Theorem 1.3, we would like to apply (3) to $(\lambda, \mu) \in \mathbb{Q}^2$. However, we find that formula (3) no longer defines a group action if we allow non-integral λ and μ , since

$$\begin{aligned} & (\phi|_m[\lambda \ \mu])|_m[\lambda' \ \mu'])(\tau, z) = \\ &= e^m(\lambda'^2\tau + 2\lambda'z + \lambda^2\tau + 2\lambda(z + \lambda'\tau + \mu'))\phi(\tau, z + \lambda'\tau + \mu' + \lambda z + \mu) \\ &= e(2m\lambda\mu')(\phi|_m[\lambda+\lambda' \ \mu+\mu']) (\tau, z) \end{aligned}$$

and $e(2m\lambda'\mu)$ will not in general be equal to 1. Similarly, the third equation of (4) breaks down if X is not in \mathbb{Z}^2 . Hence if we want to extend our actions to $SL_2(\mathbb{Q})$ (or $SL_2(\mathbb{R})$) and \mathbb{Q}^2 (or \mathbb{R}^2), we must modify the definition of the group action.

The verification of the third equation in (4) depends on the two

elementary identities

$$\frac{z}{c\tau+d} + \lambda \frac{a\tau+b}{c\tau+d} + \mu = \frac{z + \lambda_1\tau + \mu_1}{c\tau+d} ,$$

$$\lambda^2 \frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d} + \lambda\mu = \lambda_1^2\tau + 2\lambda_1 z - \frac{c(z + \lambda_1\tau + \mu_1)^2}{c\tau+d} + \lambda_1$$

where $(\lambda_1 \mu_1) = (\lambda \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Thus to make this equation hold for arbitrary $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $X = (\lambda \mu) \in \mathbb{R}^2$ we should replace (3) by

$$(5) \quad (\phi|_m[\lambda \mu])(\tau, z) := e^M(\lambda^2\tau + 2\lambda z + \lambda\mu)\phi(\tau, z + \lambda\tau + \mu) \quad ((\lambda \mu) \in \mathbb{R}^2) ;$$

this is compatible with (3) because $e^M(\lambda\mu) = 1$ for $\lambda, \mu \in \mathbb{Z}$.

Unfortunately, (5) still does not define a group action; we now find

$$(6) \quad (\phi|_m X)|_m X' = e^M(\lambda\mu' - \lambda'\mu)\phi|_m(X + X') \quad (X = (\lambda \mu), \quad X' = (\lambda' \mu') \in \mathbb{R}^2) .$$

To absorb the extra factor, we must introduce a scalar action of the group \mathbb{R} by

$$(7) \quad (\phi|_m[\kappa])(\tau, z) := e(m\kappa)\phi(\tau, z) \quad (\kappa \in \mathbb{R})$$

and then make a central extension of \mathbb{R}^2 by this group \mathbb{R} ; i.e. replace \mathbb{R}^2 by the Heisenberg group

$$H_{\mathbb{R}} := \{[(\lambda \mu), \kappa] \mid (\lambda, \mu) \in \mathbb{R}^2, \kappa \in \mathbb{R}\} ,$$

$$[(\lambda \mu), \kappa][(\lambda' \mu'), \kappa'] = [(\lambda + \lambda' \mu + \mu'), \kappa + \kappa' + \lambda\mu' - \lambda'\mu] .$$

(This group is isomorphic to the group of upper triangular unipotent 3×3 matrices via

$$[(\lambda \mu), \kappa] \mapsto \begin{pmatrix} 1 & \lambda & \frac{1}{2}(\kappa + \lambda\mu) \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix} .)$$

The subgroup $C_{\mathbb{R}} := \{(0, 0), \kappa\}, \kappa \in \mathbb{R}\}$ is the center of $H_{\mathbb{R}}$ and $H_{\mathbb{R}}/C_{\mathbb{R}} \cong \mathbb{R}^2$. We can now combine (5) and (7) into an action of $H_{\mathbb{R}}$ by setting

$$(\phi | [(\lambda \mu), \kappa])(\tau, z) = e^{i\kappa} (\lambda^2 \tau + 2\lambda z + \lambda\mu + \kappa) \phi(\tau, z + \lambda\tau + \mu) ,$$

and this now is a group action because the extra factor $e^{i\kappa} (\lambda' \mu - \lambda \mu')$ in (6) is compensated by the twisted group law in $H_{\mathbb{R}}$. Because this twist involves $\lambda \mu' - \lambda' \mu = \det \begin{pmatrix} \lambda & \mu \\ \lambda' & \mu' \end{pmatrix}$ and the determinant is preserved by SL_2 , the group $SL_2(\mathbb{R})$ acts on $H_{\mathbb{R}}$ on the right by

$$[X, \kappa]M = [XM, \kappa] \quad (X \in \mathbb{R}^2, \kappa \in \mathbb{R}, M \in SL_2(\mathbb{R})) ;$$

the above calculations then show that all three identities (4) remain true if we now take $M, M' \in SL_2(\mathbb{R})$ and $X, X' \in H_{\mathbb{R}}$ and hence that equations (2), (5) and (7) together define an action of the semidirect product $SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}$.

In the situation of usual modular forms, we write \mathcal{H} as G/K , where $G = SL_2(\mathbb{R})$ contains Γ as a discrete subgroup with $Vol(\Gamma \backslash G)$ finite and $K = SO(2)$ is a maximal compact subgroup of G . Here we would like to do the same. However, the group $SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}$ contains $\Gamma^J = \Gamma \times \mathbb{Z}^2$ with infinite covolume (because of the extra \mathbb{R} in $H_{\mathbb{R}}$) and its quotient by the maximal compact subgroup $SO(2)$ is $\mathcal{H} \times \mathbb{C} \times \mathbb{R}$ rather than $\mathcal{H} \times \mathbb{C}$. To correct this, we observe that the subgroup $\mathbb{Z} \subset \mathbb{R}$ acts trivially in (7), so that (2), (5) and (7) actually define an action of the quotient group

$$G^J := SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}/C_{\mathbb{Z}} .$$

Here it does not matter on which side $H_{\mathbb{R}}$ we write $C_{\mathbb{Z}}$, since C is central in H ; the quotient $H_{\mathbb{R}}/C_{\mathbb{Z}}$ is a central extension of \mathbb{R}^2 by $S^1 = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$ ($\zeta = e(\kappa)$) and will also be denoted $\mathbb{R}^2 \cdot S^1$. Now Γ^J is a discrete subgroup of G with $\text{Vol}(\Gamma^J \backslash G^J) < \infty$, and if we choose the maximal compact subgroup

$$K^J := SO(2) \times S^1 \subset G^J = \text{SL}_2(\mathbb{R}) \ltimes (\mathbb{R}^2 \cdot S^1)$$

then G^J/K^J can be identified naturally with $\mathcal{H} \times \mathbb{C}$ via

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda \ \mu), \zeta \right] K^J \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{\lambda\tau + \mu}{c\tau + d}, \zeta \right).$$

The above discussion now gives

THEOREM 1.4. Let G^J be the set of triples $[M, X, \zeta]$ ($M \in \text{SL}_2(\mathbb{R})$, $X \in \mathbb{R}^2$, $\zeta \in \mathbb{C}$, $|\zeta| = 1$). Then G^J is a group via

$$[M, X, \zeta][M', X', \zeta'] = [MM', XM' + X', \zeta\zeta' \cdot e(\det \begin{pmatrix} XM' \\ X' \end{pmatrix})]$$

and the formula

$$\begin{aligned} \phi | \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda \ \mu), \zeta \right] (\tau, z) \\ = \zeta^m (c\tau + d)^{-k} e^m \left(-\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\mu \right) \\ \times \phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right) \end{aligned}$$

defines an action of G^J on $\{\phi: \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C}\}$. The functions ϕ satisfying the transformation law i) and ii) of Jacobi forms are precisely those invariant with respect to this action under the discrete subgroup $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$ of G^J , and the space of such ϕ can be identified via

$$F(g) := (\phi | g)(i, 0)$$

with the set of functions $F: G^J \rightarrow \mathbb{C}$ left invariant under Γ^J and transforming on the right by the representation

$$F\left(g \cdot \left[\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, (0, 0), \zeta \right]\right) = \zeta^m e^{ik\theta} F(g)$$

of the maximal compact subgroup $K^J = SO(2) \times S^1$ of G^J .

Thus the two integers k and m in the definition of Jacobi forms appear, as they should, as the parameters for the irreducible (and here one-dimensional) representations of a maximal compact subgroup of G^J .

As an application of all this formalism, we now give the proof of 1.3. The function $f(\tau)$ in that theorem is up to a constant (namely $e^m(\lambda\mu)$) equal to $\phi_X(\tau) := (\phi|X)(\tau, 0)$, where $X = (\lambda \mu) \in \mathbb{Q}^2$ and $\phi|X$ is defined by (5) (from now on we often omit the indices k, m on the sign $|$). For $X' = (\lambda' \mu') \in \mathbb{Z}^2$ we have

$$\phi_{X+X'}(\tau) = e^{m(\lambda\mu' - \lambda'\mu)} \phi_X(\tau)$$

by (6), so ϕ_X depends up to a scalar factor only on $X \pmod{\mathbb{Z}^2}$ and ϕ_X itself depends only on $X \pmod{N\mathbb{Z}^2}$ if $X \in N^{-1}\mathbb{Z}^2$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$\begin{aligned} (c\tau + d)^{-k} \phi_X \left(\frac{a\tau + b}{c\tau + d} \right) &= (\phi|X|M)(\tau, 0) \\ &= (\phi|M|(XM))(\tau, 0) \\ &= (\phi|(XM))(\tau, 0) \\ &= \phi_{XM}(\tau), \end{aligned}$$

so ϕ_X behaves like a modular form with respect to the congruence subgroup

$$\{M \in \Gamma \mid XM \equiv X \pmod{\mathbb{Z}^2}, m \cdot \det \begin{pmatrix} X \\ XM \end{pmatrix} \in \mathbb{Z}\}$$

of Γ (this group can be written explicitly

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid (a-1)\lambda + c\mu, b\lambda + (d-1)\mu, m(c\mu^2 + (d-a)\lambda\mu - b\lambda^2) \in \mathbb{Z} \right\}$$

and hence contains $\Gamma \cap \Gamma\left(\frac{N^2}{(N,m)}\right)$ if $NX \in \mathbb{Z}^2$. Finally, if M is any element of Γ_1 then

$$(\phi|_M)_k(\tau) = (\phi|M|XM)(\tau, 0) = e^m(\lambda_1^2\tau + \lambda_1\mu_1)(\phi|M)(\tau, \lambda_1\tau + \mu_1)$$

where $(\lambda_1\mu_1) = XM$, and since $\phi|M$ has a Fourier development containing $q^n\zeta^r$ only for $4nm \geq r^2$, this contains only nonnegative powers of $e(\tau)$ by the same calculation as given for $M = \text{Id}$ after the statement of 1.3.

We end with one other simple, but basic, property of Jacobi forms

THEOREM 1.5. *The Jacobi forms form a bigraded ring.*

Proof. That the product of two Jacobi forms ϕ_1 and ϕ_2 of weight k_1 and k_2 and index m_1 and m_2 , respectively, transforms like a Jacobi form of weight $k = k_1 + k_2$ and index $m = m_1 + m_2$ is clear; we have to check the condition at infinity. One way to see this is to use the converse of Theorem 1.3, i.e. to observe that the condition at infinity for a Jacobi form $\phi(\tau, z)$ of index m is equivalent to the condition that $f(\tau) = e^m(\lambda^2\tau)\phi(\lambda\tau + \mu)$ be holomorphic at ∞ (in the usual sense) for all $\lambda, \mu \in \mathbb{Q}$; this condition is clearly satisfied for $\phi(\tau, z) = \phi_1(\tau, z)\phi_2(\tau, z)$ with $f(\tau) = f_1(\tau)f_2(\tau)$. A more direct proof is to write the (n, r) -Fourier coefficient of ϕ as

$$c(n, r) = \sum_{\substack{n_1+n_2=n \\ r_1+r_2=r}} c_1(n_1, r_1)c_2(n_2, r_2) ,$$

where the c_i are the Fourier coefficients of ϕ_i (the sum is finite since $n_i \leq n$, $r_i^2 \leq 4n_i m_i$) and deduce the inequality $r^2 \leq 4nm$ from the identity

$$n_1+n_2 - \frac{(r_1+r_2)^2}{4(m_1+m_2)} = \left(n_1 - \frac{r_1^2}{4m_1}\right) + \left(n_2 - \frac{r_2^2}{4m_2}\right) + \frac{(m_1r_2 - m_2r_1)^2}{4m_1m_2(m_1+m_2)} .$$

This identity also shows that (as for modular forms) the product $\phi_1\phi_2$ is a cusp form whenever ϕ_1 or ϕ_2 is one but that (unlike the situation for modular forms) $\phi_1\phi_2$ can be a cusp form even if neither ϕ_1 nor ϕ_2 is.

The ring $J_{*,*} = \bigoplus_{k,m} J_{k,m}$ of Jacobi forms will be the object of study of Chapter III.

§2. Eisenstein Series and Cusp Forms

As in the usual theory of modular forms, we will obtain our first examples of Jacobi forms by constructing Eisenstein series. In the modular case one sets (for $k > 2$)

$$E_k(\tau) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_1} 1|_k = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} (c\tau + d)^{-k},$$

where $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} | n \in \mathbb{Z}\}$ is the subgroup of Γ_1 of elements γ with $1|_k = 1$, where 1 denotes the constant function. Similarly, here we define

$$(1) \quad E_{k,m}(\tau, z) := \sum_{\gamma \in \Gamma_\infty^J \setminus \Gamma_1^J} 1|_{k,m},$$

where

$$\begin{aligned} \Gamma_\infty^J &= \{\gamma \in \Gamma_1^J | 1|\gamma = 1\} \\ &= \{[\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0 \ \mu)] \mid n, \mu \in \mathbb{Z}\} \quad . \end{aligned}$$

Explicitly, this is

$$(2) \quad E_{k,m}(\tau, z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} e^m \left(\lambda^2 \frac{a\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right)$$

where a, b are chosen so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$. As in the case of modular forms, the series converges absolutely for $k \geq 4$; it is zero if k is odd (replace c, d by $-c, -d$). The invariance of $E_{k,m}$ under Γ^J is clear from the definition and the absolute convergence. To check the cusp condition, and in order to have an explicit example of a form in $J_{k,m}$, we must calculate the Fourier development of $E_{k,m}$, which we now proceed to do.

As with E_k , we split the sum over c, d into two parts, according as c is 0 or not. If $c=0$, then $d=\pm 1$; these terms give a contributi

$$(3) \quad \sum_{\lambda \in \mathbb{Z}} e^{\pi i(\lambda^2 \tau + 2\lambda z)} = \sum_{\lambda \in \mathbb{Z}} q^{m\lambda^2} \zeta^{2m\lambda}$$

($q = e^{2\pi i \tau}$, $\zeta = e^{2\pi i z}$). This is a linear combination of $q^n \zeta^r$ with $4nm = r^2$ and corresponds to the constant term of the usual Eisenstein series. If $c \neq 0$, we can assume $c > 0$ (since k is even); using the identity

$$\frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d} = -\frac{c(z-\lambda/c)^2}{c\tau+d} + \frac{a\lambda^2}{c} \quad (c \neq 0)$$

we can write these terms as

$$\sum_{c=1}^{\infty} c^{-k} \sum_{\substack{d \in \mathbb{Z} \\ (d,c)=1}} \sum_{\lambda \in \mathbb{Z}} \left(\tau + \frac{d}{c} \right)^{-k} e^{\pi i \left(-\frac{(z-\lambda/c)^2}{\tau+d/c} + \frac{a\lambda^2}{c} \right)} .$$

Note that $d \mapsto d+c$ and $\lambda \mapsto \lambda+c$ correspond to $z \mapsto z+1$ and $\tau \mapsto \tau+1$, so this part equals

$$(4) \quad \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{d \pmod{c} \\ (d,c)=1}} \sum_{\lambda \pmod{c}} e_c(m d^{-1} \lambda^2) F_{k,m}\left(\tau + \frac{d}{c}, z - \frac{\lambda}{c}\right)$$

with e_c as in "Notations" and

$$F_{k,m}(\tau, z) := \sum_{p,q \in \mathbb{Z}} (\tau + p)^{-k} e^m \left(-\frac{(z+q)^2}{\tau + p} \right);$$

the function $F_{k,m}$ is periodic in τ and z , so (4) makes sense. Now the usual Poisson summation formula gives

$$F_{k,m} = \sum_{n,r \in \mathbb{Z}} \gamma(n,r) q^n \zeta^r$$

with

$$\gamma(n,r) = \int_{\text{Im}(\tau)=C_1} \tau^{-k} e(-n\tau) \int_{\text{Im}(z)=C_2} e(-mz^2/\tau - rz) dz d\tau$$

($C_1 > 0$, C_2 arbitrary). The inner integral is standard and equals $(\tau/2im)^{\frac{1-k}{2}} e(r^2\tau/4m)$. Hence

$$\begin{aligned} \gamma(n,r) &= \int_{\text{Im}(\tau)=C_1} \tau^{-k} (\tau/2im)^{\frac{1-k}{2}} e\left(\frac{r^2 - 4nm}{4m}\tau\right) d\tau \\ &= \begin{cases} 0 & \text{if } r^2 \geq 4nm \\ \alpha_k m^{1-k} (4nm - r^2)^{k-\frac{3}{2}} & \text{if } r^2 < 4nm \end{cases} \end{aligned}$$

with

$$\alpha_k := \frac{(-1)^{k/2} \pi^{k-\frac{1}{2}}}{2^{k-2} \Gamma(k-\frac{1}{2})}$$

(if $r^2 \geq 4nm$, we can deform the path of integration to $+i\infty$, so $\gamma = 0$; if $r^2 < 4nm$, we deform it to a path from $-i\infty$ to $-i\infty$ circling 0 once in a clockwise direction and obtain a standard integral representation of $1/\Gamma(s)$). Substituting the Fourier development of $F_{k,m}$ into (4) gives the expression

$$\sum_{\substack{n,r \in \mathbb{Z} \\ 4nm > r^2}} e_{k,m}(n,r) q^n \zeta^r$$

with

$$(5) \quad e_{k,m}(n,r) = \frac{\alpha_k}{m^{k-1}} (4nm - r^2)^{k-3/2} \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{\lambda, d \pmod{c} \\ (d,c)=1}} e_c(m\lambda^{-1}\lambda^2 - r\lambda + nd)$$

(for d^{-1} , see "Notations"). To calculate this, we first replace λ by $d\lambda$ in the inner double sum (since $(d,c)=1$, this simply permutes the summands); then the summand becomes $e_c(dQ(\lambda))$ with $Q(\lambda) := m\lambda^2 + r\lambda + n$. We now use the well-known identity

$$\sum_{\substack{d \pmod{c} \\ (d,c)=1}} e_c(dN) = \sum_{a|c} \mu\left(\frac{c}{a}\right) a ,$$

where μ is the Möbius function (so-called Ramanujan sum; see Hardy-Wright or most other number theory texts); then the inner double sum in (5) becomes

$$\sum_{a|c} \mu\left(\frac{c}{a}\right) a \sum_{\substack{\lambda \pmod{c} \\ Q(\lambda) \equiv 0 \pmod{a}}} 1 .$$

Now the condition $Q(\lambda) \equiv 0 \pmod{a}$ depends only on $\lambda \pmod{a}$, so the inner sum is $\frac{c}{a}$ times $N_a(Q)$, where

$$N_a(Q) := \#\{\lambda \pmod{a} \mid Q(\lambda) \equiv 0 \pmod{a}\} .$$

Hence the triple sum in (5) simplifies to

$$\sum_{c=1}^{\infty} c^{1-k} \sum_{a|c} \mu\left(\frac{c}{a}\right) N_a(Q) = \zeta(k-1)^{-1} \sum_{a=1}^{\infty} \frac{N_a(Q)}{a^{k-1}}$$

(the last equality follows by writing $c = ab$ and using $\sum \mu(b)b^{-s} = \zeta(s)^{-1}$). To calculate the Dirichlet series, we first calculate $N_a(Q)$ for $(a,m)=1$; this will suffice completely if $m=1$ and (using the obvious multiplicativity of N_a) will give the Dirichlet series up to a finite Euler product involving the prime divisors of m in general. If $(a,m)=1$,

then

$$\begin{aligned} N_a(Q) &= \#\{\lambda \pmod{a} \mid m\lambda^2 + r\lambda + n \equiv 0 \pmod{a}\} \\ &= \#\{\lambda \pmod{a} \mid (2m\lambda + r)^2 \equiv r^2 - 4nm \pmod{4a}\} \\ &= N_a(r^2 - 4nm), \end{aligned}$$

where

$$N_a(D) := \#\{x \pmod{2a} \mid x^2 \equiv D \pmod{4a}\}.$$

It is a classical fact that

$$(6) \quad \sum_{a=1}^{\infty} N_a(D) a^{-s} = \frac{\zeta(s)}{\zeta(2s)} L_D(s),$$

if $D=1$ or if D is the discriminant of a real quadratic field,
 where $L_D(s) \approx L(s, \frac{D}{\cdot})$ is the Dirichlet L-series associated to D .
 It was shown in [39, p.130] that the same formula holds for all $D \in \mathbb{Z}$
 if $L_D(s)$ is defined by

$$L_D(s) = \begin{cases} 0 & \text{if } D \not\equiv 0, 1 \pmod{4}, \\ \zeta(2s-1) & \text{if } D = 0, \\ L_{D_0}(s) \cdot \sum_{d|f} \mu(d) \left(\frac{D_0}{d}\right) d^{-s} \sigma_{1-2s}(f/d) & \text{if } D \equiv 0, 1 \pmod{4}, D \neq 0 \end{cases}$$

where in the last line D has been written as $D_0 f^2$ with $f \in \mathbb{N}$ and
 $D_0 = \text{discriminant of } \mathbb{Q}(\sqrt{D})$ (the finite sum in this case can also be
 written as a finite Euler product over the prime divisors of f).

Inserting (6) into the preceding equations, we find that we have proved

$$e_{k,1}(n, r) = \alpha_k |D|^{k-\gamma_2} \zeta(2k-2)^{-1} L_D(k-1)$$

if $m=1$ and $D = r^2 - 4n < 0$, while for m arbitrary there is a similar
 formula (now with $D = r^2 - 4nm$) but multiplied by an Euler factor

involving the prime divisors of m . Using the functional equations of $L_D(s)$ and $\zeta(s)$ we can rewrite this formula in the simpler form

$$e_{k,1}(n,r) = L_D(2-k)/\zeta(3-2k) ,$$

where now all numerical factors have disappeared. The values $L_D(2-k)$ ($D < 0$, k even) are well-known to be rational and non-zero; they have been studied extensively by Cohen [6], who denoted them $H(k-1, |D|)$. Summarizing, we have proved

THEOREM 2.1. *The series $E_{k,m}$ ($k \geq 4$ even) converges and defines a non-zero element of $J_{k,m}$. The Fourier development of $E_{k,m}$ is given by*

$$E_{k,m}(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4nm \geq r^2}} e_{k,m}(n,r) q^n \zeta^r$$

where $e_{k,m}(n,r)$ for $4nm = r^2$ equals 1 if $r \equiv 0 \pmod{2m}$ and 0 otherwise, while for $4nm > r^2$ we have

$$e_{k,1}(n,r) = \frac{H(k-1, \frac{4nm-r^2}{4})}{\zeta(3-2k)}$$

($H(k-1, N) = L_{-N}(2-k)$ = Cohen's function) and

$$e_{k,m}(n,r) = \frac{H(k-1, \frac{4nm-r^2}{4})}{\zeta(3-2k)} \cdot \prod_{p|m} \text{(elementary p-factor)} .$$

In particular, $e_{k,m}(n,r) \in \mathbb{Q}$.

One can in fact complete the calculation of $e_{k,m}$ in general with little extra work; the result for m square-free is

$$(7) \quad e_{k,m}(n,r) = \frac{\sigma_{k-1}(m)^{-1}}{\zeta(3-2k)} \sum_{d|(n,r,m)} d^{k-1} H\left(k-1, \frac{4nm-r^2}{d^2}\right) .$$

However, we do not bother to give the calculation since this result will follow from the properties of Hecke-type operators introduced in §4 (Theorem 4.3).

For $m=1$ and the first few values of k we find, using the tables of $H(k-1, N)$ given in [6], the expansions

$$\begin{aligned} E_{4,1} &= 1 + (\zeta^2 + 56\zeta + 126 + 56\zeta^{-1} + \zeta^{-2})q \\ &+ (126\zeta^2 + 576\zeta + 756 + 576\zeta^{-1} + 126\zeta^{-2})q^2 \\ &+ (56\zeta^3 + 756\zeta^2 + 1512\zeta + 2072 + 1512\zeta^{-1} + 756\zeta^{-2} + 56\zeta^{-3})q^3 + \dots \end{aligned}$$

$$\begin{aligned} E_{6,1} &= 1 + (\zeta^2 - 88\zeta - 330 - 88\zeta^{-1} + \zeta^{-2})q \\ &+ (-330\zeta^2 - 4224\zeta - 7524 - 4224\zeta^{-1} - 330\zeta^{-2})q^2 + \dots , \end{aligned}$$

$$E_{8,1} = 1 + (\zeta^2 + 56\zeta + 366 + 56\zeta^{-1} + \zeta^{-2})q^2 + \dots .$$

Further coefficients of these and other Jacobi forms of index 1 are given in the tables on pp.141-143.

In the formula for the Fourier coefficients of $E_{k,1}$, it is striking that $e_{k,1}(n,r)$ depends only on $4n-r^2$. We now show that this is true for any Jacobi form of index 1; more generally, we have

THEOREM 2.2. *Let ϕ be a Jacobi form of index m with Fourier development $\sum c(n,r)q^n\zeta^r$. Then $c(n,r)$ depends only on $4nm-r^2$ and on $r(\text{mod } 2m)$. If k is even and $m=1$ or m is prime, then $c(n,r)$ depends only on $4nm-r^2$. If $m=1$ and k is odd, then ϕ is identically zero.*

Proof. This is essentially a restatement of the second transformation law of Jacobi forms: we have

$$\begin{aligned}\sum c(n, r) q^n \zeta^r &= \phi(\tau, z) = e^m (\lambda^2 \tau + 2\lambda z) \phi(\tau, z + \lambda \tau + \mu) \\ &= q^{m\lambda^2} \zeta^{2m\lambda} \sum c(n, r) q^{n+r\lambda+m\lambda^2} \zeta^{r+2m\lambda} \\ &= \sum c(n, r) q^{n+r\lambda+m\lambda^2} \zeta^{r+2m\lambda}\end{aligned}$$

and hence

$$c(n, r) = c(n+r\lambda+m\lambda^2, r+2m\lambda),$$

i.e. $c(n, r) = c(n', r')$ whenever $r' \equiv r \pmod{2m}$ and $4n'm - r'^2 = 4nm - r^2$ as stated in the theorem. If k is even, then we also have

$c(n, -r) = c(n, r)$ (because applying the first transformation law of Jacobi forms to $-I_2 \in \Gamma_1$ gives $\phi(\tau, -z) = (-1)^k \phi(\tau, z)$), so if m is 1 or a prime, then

$$4n'm - r'^2 = 4nm - r^2 \Rightarrow r' \equiv \pm r \pmod{2m} \Rightarrow c(n, r) = c(n', r').$$

Finally, if $m=1$ and k is odd then $\phi \equiv 0$ because $c(n, -r) = -c(n, r)$ but $4nm - (-r)^2 = 4nm - r^2$ and $-r \equiv r \pmod{2m}$ in this case.

Remark: Theorem 2.2 is the basis of the relationship between Jacobi forms and modular forms of half-integral weight (cf. §5).

In the definition of Jacobi cusp forms, there were apparently infinitely many conditions to check, namely $c(n, r) = 0$ for all n, r with $4nm = r^2$. Theorem 2.2 tells us in particular that we in fact need only check this for a set of representatives of $r \pmod{2m}$. The number of residue classes $r \pmod{2m}$ with $r^2 \equiv 0 \pmod{4m}$ is b , where b^2 is the largest square dividing m (namely if $m = ab^2$ with a square-free, then $4m|r^2 \Leftrightarrow 2ab|r$). Thus for $\phi \in J_{k,m}$ we have

$$\phi \text{ a cusp form} \Leftrightarrow c(as^2, 2abs) = 0 \text{ for } s = 0, 1, \dots, b-1;$$

in particular, the codimension of $J_{k,m}^{\text{cusp}}$ in $J_{k,m}$ is at most b . Using

$c(n, -r) = (-1)^k c(n, r)$ we see that in fact it suffices to check the condition $c(as^2, 2abs) = 0$ for $s = 0, 1, \dots, \lfloor \frac{b}{2} \rfloor$ if k is even and $s = 1, 2, \dots, \lfloor \frac{b-1}{2} \rfloor$ if k is odd. Hence we have

THEOREM 2.3. *The codimension of $J_{k,m}^{\text{cusp}}$ in $J_{k,m}$ is at most $\lfloor \frac{b}{2} \rfloor + 1$ if k is even (resp. $\lfloor \frac{b-1}{2} \rfloor$ if k is odd), where b is the largest integer such that $b^2|m$.*

On the other hand, if $k > 2$ then for each integer s we can construct an Eisenstein series

$$(8) \quad E_{k,m,s}(\tau, z) := \sum_{\gamma \in \Gamma_{\infty}^J \setminus \Gamma^J} q^{as^2} \zeta^{2abs} |\gamma$$

($m = ab^2$ as above), where the summation is the same as in the definition of $E_{k,m} = E_{k,m,0}$. Then repeating the beginning of the proof of Theorem 2.1 we find that

$$(9) \quad E_{k,m,s} = \sum_{\substack{r \in \mathbb{Z} \\ r \equiv 2abs \pmod{2m}}} q^{r^2/4m} (\zeta^r + (-1)^k \zeta^{-r}) + \dots ,$$

where "... " (the contribution from all terms in the sum with $c \neq 0$) has a Fourier development consisting only of terms $q^n \zeta^r$ with $4nm - r^2 > 0$. It is then clear that $E_{k,m,s}$ depends only on $s \pmod{b}$, that $E_{k,m,-s} = (-1)^k E_{k,m,s}$, and that the series $E_{k,m,s}$ with $0 \leq s \leq \frac{b}{2}$ (k even) or $0 < s < \frac{b}{2}$ (k odd) are linearly independent. Comparing this with 2.3, we see that the bound given there is sharp and that we have proved:

THEOREM 2.4. *If $k > 2$, then $J_{k,m} = J_{k,m}^{\text{cusp}} \oplus J_{k,m}^{\text{Eis}}$, where $J_{k,m}^{\text{cusp}}$ is the space of cusp forms in $J_{k,m}$ and $J_{k,m}^{\text{Eis}}$ is the space spanned by*

the functions $E_{k,m,s}$. The functions $E_{k,m,s}$ with $0 \leq s \leq \frac{b}{2}$ (k even) or $0 < s < \frac{b}{2}$ (k odd) form a basis $J_{k,m}^{Eis}$.

We will not give the entire calculation of the Fourier development of the functions $E_{k,m,s}$ here, since it is tedious and we do not need the result. However, we make some remarks. In §4 we will introduce certain operators U_ℓ and V_ℓ which map Jacobi forms to Jacobi forms of higher index. These will act in a simple way on Fourier developments and will send Eisenstein series to Eisenstein series. Hence certain combinations of the $E_{k,m,s}$ ("old forms") have Fourier coefficients which can be given in a simple way in terms of the Fourier coefficients of Eisenstein series of lower index (compare equation (7), where the coefficients of $E_{k,m}$ are simple linear combinations of those of $E_{k,1}$), and we need only consider the remaining, "new", forms. A convenient basis for these is the set of forms

$$(10) \quad E_{k,m}^{(X)} := \sum_{s \pmod f} X(s) E_{k,m,s} \quad (m = f^2)$$

of index f^2 , where X is a primitive Dirichlet character $(\text{mod } f)$ with $X(-1) = (-1)^k$. Then a calculation analogous to the proof of Theorem 2.1 for the case $m=1$ shows that the coefficient $q^n \zeta^r$ in $E_{k,m}^{(X)}$ is given by

$$(11) \quad e_{k,m}^{(X)}(n,r) = e(X) X(r) L_{r^2 - 4nm}(2-k, \bar{X})$$

if $(r,f)=1$, where $L_D(s,X)$ is the convolution of $L_D(s)$ and $L(s,X)$ and $e(X)$ a simple constant (essentially a quotient of Gauss sums attached to X and X^2 divided by $L(3-2k, X^{-2})$); in particular, the coefficients are algebraic (in $\mathbb{Q}(X)$) and non-zero. If $(r,f) > 1$, then $e_{k,m}^{(X)}(n,r)$ is given by a formula like (11) with the right-hand side multiplied by a finite Euler product extending over the common prime

factors of r and f .

If $k = 2$, then the Eisenstein series fail to converge; however, by the same type of methods as are used for ordinary modular forms ("Hecke's convergence trick") one can show that for X non-principal there is an Eisenstein series $E_{2,m,X} \in J_{2,m}$ having a Fourier development given by the same formula as for $k > 2$. Since X must be even ($\chi(-1) = (-1)^k$) and since there exists an even non-principal character $(\bmod b)$ only if $b = 5$ or $b \geq 7$, such series exist only for m divisible by 25, 49, 64,

There is one more topic from the theory of cusp forms in the classical case which we want to generalize, namely the characterization of cusp forms in terms of the Petersson scalar product. We write

$$\tau = u + iv \quad (v > 0), \quad z = x + iy$$

and define a volume element dV on $\mathcal{H} \times \mathbb{C}$ by

$$(12) \quad dV := v^{-3} dx dy du dv .$$

It is easily checked that this is invariant under the action of G^J on $\mathcal{H} \times \mathbb{C}$ defined in §1 and is the unique G^J -invariant measure up to a constant. (The form $v^{-2} du dv$ is the usual $SL_2(\mathbb{R})$ -invariant volume form on \mathcal{H} ; the form $v^{-1} dx dy$ is the translation-invariant volume form on \mathbb{C} , normalized so that the fibre $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ has volume 1.) If ϕ and ψ transform like Jacobi forms of weight k and index m , then the expression

$$v^k e^{-4\pi my^2/v} \phi(\tau, z) \overline{\psi(\tau, z)}$$

is easily checked to be invariant under Γ^J , so we can define the Petersson scalar product of ϕ and ψ by

$$(13) \quad (\phi, \psi) := \int_{\Gamma^J \backslash \mathcal{H} \times \mathbb{C}} v^k e^{-4\pi my^2/v} \phi(\tau, z) \overline{\psi(\tau, z)} dV .$$

Then we have

THEOREM 2.5. *The scalar product (13) is well defined and finite for $\phi, \psi \in J_{k,m}$ and at least one of ϕ and ψ a cusp form. It is positive-definite on $J_{k,m}^{\text{cusp}}$, and the orthogonal complement of $J_{k,m}^{\text{cusp}}$ with respect to $(,)$ is $J_{k,m}^{\text{Eis}}$.*

This will follow from the results in §5 concerning the connection between Jacobi forms and modular forms of half-integral weight.

§3. Taylor Expansions of Jacobi Forms

The restriction of a Jacobi form $\phi(\tau, z)$ to $z=0$ gives a modular form of the same weight. In §1 we proved an analogous statement for the restriction to $z = \lambda\tau + \mu$ (λ, μ rational) and used it to show that $J_{k,m}(\Gamma)$ is finite-dimensional. Another and even more useful way to get modular forms is to consider the Taylor development of ϕ around $z=0$; by forming certain linear combinations of the coefficients one obtains a series of modular forms $\mathcal{D}_v \phi$ (\mathcal{D}_v for " v^{th} development coefficient") with $\mathcal{D}_0 \phi = \phi(\tau, 0)$ and $\mathcal{D}_v \phi$ a modular form of weight $k+v$. The precise result is

THEOREM 3.1. *For $v \in \mathbb{N}_0$, $k \in \mathbb{N}$ define a homogeneous polynomial $p_{2v}^{(k-1)}$ of two variables by*

$$(1) \quad \frac{(k+v-2)!}{(2v)!(k-2)!} p_{2v}^{(k-1)}(r, n) = \text{coefficient of } t^{2v} \text{ in } (1 - rt + nt^2)^{-k+1}$$

Then for $\phi \in J_{k,m}(\Gamma)$ a Jacobi form with Fourier development

$$\sum_{n,r} c(n,r) q^n \zeta^r,$$
the function

$$(2) \quad \mathcal{D}_{2v}\phi := \sum_{n=0}^{\infty} \left(\sum_r p_{2v}^{(k-1)}(r, nm) c(n, r) \right) q^n$$

is a modular form of weight $k+2v$ on Γ . If $v > 0$, it is a cusp form.

Explicitly, one has

$$\mathcal{D}_0\phi = \sum_n \left(\sum_r c(n, r) \right) q^n ,$$

$$\mathcal{D}_2\phi = \sum_n \left(\sum_r (kr^2 - 2nm)c(n, r) \right) q^n ,$$

$$\mathcal{D}_4\phi = \sum_n \left(\sum_r ((k+1)(k+2)r^4 - 12(k+1)r^2nm + 12n^2m^2)c(n, r) \right) q^n$$

Notice that the summation over r is finite since $c(n, r) \neq 0 \Rightarrow r^2 \leq 4nm$.

The polynomial $p_{2v}^{(k-1)}$ is given explicitly by

$$(3) \quad p_{2v}^{(k-1)}(r, n) := \sum_{\mu=0}^v (-1)^\mu \frac{(2v)!}{\mu!(2v-2\mu)!} \frac{(k+2v-\mu-2)}{(k+v-2)!} r^{2v-2\mu} n^\mu$$

and is, up to a change of notation and normalization, the so-called Gegenbauer or "ultraspherical" polynomial, studied in any text on orthogonal polynomials; we have chosen the normalization so as to make $p_{2v}^{(k-1)}$ a polynomial with integral coefficients in k, r, n in a minimal way (actually, $\frac{1}{v!}$ times $p_{2v}^{(k-1)}$ would still have integral coefficients as a function of r and n for fixed $k \in \mathbb{N}$). The characteristic property of the polynomial $p_{2v}^{(k-1)}$ is that the function $p_{2v}^{(k-1)}(B(x, y), Q(x)Q(y))$, where Q is a quadratic form in $2k$ variables and B the associated bilinear form, is a spherical function of x and y with respect to Q (Theorem 7.2).

There is a similar result involving odd polynomials and giving modular forms $\mathcal{D}_1\phi, \mathcal{D}_3\phi, \dots$ of weight $k+1, k+3, \dots$ (simply take $v \in \frac{1}{2} + \mathbb{N}_0$ and replace $(k+v-2)!$ by $(k+v-\frac{3}{2})!$ in (1) and (3)), but, as we shall see

this can be reduced to the even case in a trivial way, so we content ourselves with stating the latter case.

As an example of Theorem 3.1 we apply it to the function $E_{k,l}$, studied in the last section; using the formula given there for the Fourier coefficients of $E_{k,l}$ we obtain

COROLLARY (Cohen [6, Th.6.2]). Let k be even and $H(k-1,N)$ ($N \in \mathbb{N}_0$) be Cohen's function

$$H(k-1,N) = \begin{cases} L_N(2-k) & \text{if } N > 0, N \equiv 0 \text{ or } 3 \pmod{4}, \\ \zeta(3-2k) & \text{if } N = 0, \\ 0 & \text{if } N \equiv 1 \text{ or } 2 \pmod{4}. \end{cases}$$

Then for each $v \in \mathbb{N}_0$ the function

$$c_k^{(v)}(\tau) = \sum_{n \geq 0} \left(\sum_{r^2 \leq 4n} p_{2v}^{(k-1)}(r,n) H(k-1,4n-r^2) \right) q^n$$

is a modular form of weight $k+2v$ on the full modular group Γ_1 . If $v > 0$ it is a cusp form.

Cohen's proof of this result used modular forms of half-integral weight; the relation of this to Theorem 3.1 will be discussed in §5. Yet another proof was given in [39], where it was shown that $c_k^{(v)}$ has the property that its scalar product with a Hecke eigenform $f = \sum a(n)q^n \in S_{k+2v}$ is equal, up to a simple numerical factor, to the value of the Rankin series $\sum a(n)^2 n^{-s}$ at $s = 2k + 2v - 2$. This property characterizes the form $c_k^{(v)}$ and also shows (since the value of the Rankin series is non-zero) that it generates S_{k+2v} (resp. M_k if $v=0$) as a module over the Hecke algebra; an application of this will be mentioned in §7.

To prove Theorem 3.1, we first develop $\phi(\tau,z)$ in a Taylor expansion around $z = 0$:

$$(4) \quad \phi(\tau, z) = \sum_{v=0}^{\infty} \chi_v(\tau) z^v$$

and then apply the transformation equation

$$(5) \quad \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{m\left(\frac{cz^2}{c\tau+d}\right)} \phi(\tau, z)$$

to get

$$(6) \quad \chi_v\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+v} \left(\chi_v(\tau) + \frac{2\pi i m c}{c\tau+d} \chi_{v-2}(\tau) + \frac{1}{2!} \left(\frac{2\pi i m c}{c\tau+d}\right)^2 \chi_{v-4}(\tau) + \dots \right)$$

i.e. χ_v transforms under Γ like a modular form of weight $k+v$ modulo corrections coming from previous coefficients. The first three cases of (6) are

$$\chi_0\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \chi_0(\tau)$$

$$\chi_1\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+1} \chi_1(\tau)$$

$$\chi_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2} \chi_2(\tau) + 2\pi i m c (c\tau+d)^{k+1} \chi_0(\tau)$$

Differentiating the first of these equations gives

$$\chi'_0\left(\frac{a\tau+b}{c\tau+d}\right) = k c (c\tau+d)^{k+1} \chi_0(\tau) + (c\tau+d)^{k+2} \chi'_0(\tau) ,$$

and subtracting a multiple of this from the third equation gives

$$\xi_2 := \chi_2 - \frac{2\pi i m}{k} \chi'_0 \in M_{k+2}(\Gamma) .$$

Proceeding in this way, we find that for each v the function

$$(7) \quad \xi_v(\tau) := \sum_{0 \leq \mu \leq \frac{v}{2}} \frac{(-2\pi i m)^\mu (k+v-\mu-2)!}{(k+v-2)! \mu!} \chi_{v-2\mu}^{(\mu)}(\tau)$$

transforms like a modular form of weight $k+v$ on Γ . The algebraic manipulations required to obtain the appropriate coefficients in (7) directly (i.e. like what we just did for $v=2$) are not very difficult

and can be made quite simple by a judicious use of generating series, but we will in fact prove the result in a slightly different way in a moment. If ϕ is periodic in z and has a Fourier development

$$\sum_{n,r} c(n,r) q^n \zeta^r, \text{ then } X_v = \frac{1}{v!} \sum_n \left(\sum_r (2\pi i r)^v c(n,r) \right) q^n \text{ and hence}$$

$$(8) \quad \xi_v(\tau) = (2\pi i)^v \sum_{n \geq 0} \left(\sum_r \left(\sum_{0 \leq \mu \leq v} \frac{(k+v-\mu-2)!}{(k+v-2)!} \frac{(-m)^{\mu} r^{v-2\mu}}{\mu! (v-2\mu)!} \right) c(n,r) \right) q^n$$

so

$$(9) \quad D_{2v}\phi(\tau) = (2\pi i)^{-2v} \frac{(k+2v-2)! (2v)!}{(k+v-2)!} \xi_{2v}(\tau).$$

Thus Theorem 3.1 follows from the following more general result:

THEOREM 3.2. *Let $\phi(\tau, z)$ be a formal power series in z as in (4) with coefficients X_v which satisfy (6) for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and are holomorphic everywhere (including the cusps of Γ). Then the function ξ_v defined by (7) is a modular form of weight $k+v$ on Γ .*

Proof. Let $M_{k,m}(\Gamma)$ denote the set of all functions ϕ satisfying the conditions of the theorem. (Note that $M_{k,m}(\Gamma)$ is isomorphic to $M_{k,1}(\Gamma)$ via $z \mapsto \sqrt{m} z$.) Since ξ_v involves only X_v , with $v' \equiv v \pmod{2}$ we can split up $M_{k,m}(\Gamma)$ into odd and even power series, say, $M_{k,m}(\Gamma) = M_{k,m}^+(\Gamma) \oplus M_{k,m}^-(\Gamma)$ and look at the two parts separately (this corresponds to adjoining $-I_2$ to Γ and looking at the action of $-I_2$ on ϕ ; if Γ already contains $-I_2$, then $M_{k,m}(\Gamma) = M_{k,m}^{(-1)^k}(\Gamma)$). If $\phi \in M_{k,m}^-(\Gamma)$, then $\phi = z\phi_1$ with $\phi_1 \in M_{k+1,m}^+(\Gamma)$ and the functions X_v, ξ_v for ϕ and ϕ_1 are the same except for the shift $v + v-1$, $k + k+1$. Hence it suffices to look at $M_{k,m}^+(\Gamma)$. We now introduce the differential operators

$$L := 8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}$$

(the heat operator) and

$$L_k := L - \frac{2k-1}{z} \frac{\partial}{\partial z} .$$

The operator L is natural in the context of Jacobi forms because it acts on monomials $q^n \zeta^r$ by multiplication by $(2\pi i)^2 (4nm - r^2)$ and hence, in view of Theorem 2.2, preserves the second transformation law of Jacobi forms; this can also be seen directly by checking that

$$(10) \quad L(\phi|_m X) = (L\phi)|_m X \quad (X \in \mathbb{R}^2) .$$

If L satisfied a similar equation with respect to the operation of $SL_2(\mathbb{R})$, then it would map Jacobi forms to Jacobi forms. Unfortunately, this is not quite true; when we compute the difference between $L(\phi|_{k,m} M)$ and $(L\phi)|_{k+2,m} M$ we find that most of the terms cancel but there is one term, $4\pi im(2k-1) \frac{c}{ct+d} (\phi|_{k,m} M)(\tau, z)$, left over (unless $k = \frac{1}{2}$, in which case L really does map Jacobi forms to Jacobi forms of weight $\frac{5}{2}$ and the same index m ; examples are the Jacobi theta-series, which are annihilated by L). To correct this we replace L by L_k , which no longer satisfies (10) but does satisfy

$$(11) \quad L_k(\phi|_{k,m} M) = (L_k\phi)|_{k+2,m} M \quad (M \in SL_2(\mathbb{R})) ,$$

as one checks by direct computation. Because of the z in the denominator, L_k acts only on power series with no linear term; in particular it acts on $M_{k,m}^+(\Gamma)$ and (because of (11)) maps $M_{k,m}^+(\Gamma)$ to $M_{k+2,m}^+(\Gamma)$. Explicitly, we have

$$L_k: \sum_{\lambda \geq 0} \chi_\lambda z^{2\lambda} \mapsto \sum_{\lambda \geq 0} (8\pi im\chi'_\lambda - 4(\lambda+1)(\lambda+k)\chi_{\lambda+1}) z^{2\lambda} .$$

Iterating this formula v times, we find by induction on v that the composite map

$$M_{k,m}^+(\Gamma) \xrightarrow{L_k} M_{k+2,m}^+(\Gamma) \xrightarrow{L_{k+2}} \dots \xrightarrow{L_{k+2v-2}} M_{k+2v,m}^+(\Gamma)$$

maps $\sum_{\lambda} X_{\lambda} z^{2\lambda}$ to

$$\sum_{\lambda \geq 0} \left(\sum_{\mu=0}^v (-4)^{v-\mu} (8\pi i m)^{\mu} \binom{v}{\mu} \frac{(\lambda+v-\mu)!}{\lambda!} \frac{(\lambda+k+2v-\mu-2)!}{(\lambda+k+v-2)!} X_{\lambda+v-\mu}^{(\mu)}(\tau) \right) z^{2\lambda},$$

and composing this with the map

$$M_{k+2v,m}^+(\Gamma) \rightarrow M_{k+2v}(\Gamma) \quad (\phi(\tau, z) \mapsto \phi(\tau, 0))$$

gives $\xi_v \in M_{k+2v}(\Gamma)$. This proves Theorem 3.2 and hence also Theorem 3.1 except for the assertion about cusp forms. But the latter is clearly true, because the constant term of (2) is $p_{2v}^{(k-1)}(0,0)c(0,0)$, which is 0 for $v > 0$, and the expansion of $D_v \phi$ at the other cusps is given by a similar formula applied to $\phi|_{k,m} M$, $M \in \Gamma_1$.

By mapping an even (resp. odd) function $\phi \in M_{k,m}(\Gamma)$ to $(\xi_0, \xi_2, \xi_4, \dots)$ (resp. to (ξ_1, ξ_3, \dots)), we obtain maps

$$M_{k,m}^+(\Gamma) \rightarrow \prod_{v \geq 0} M_{k+2v}(\Gamma),$$

$$M_{k,m}^-(\Gamma) \rightarrow \prod_{v \geq 0} M_{k+2v+1}(\Gamma).$$

It is clear that these maps are isomorphisms: one can express X_v in terms of ξ_v by inverting (7) to get

$$(12) \quad X_v(\Gamma) = \sum_{0 \leq \mu \leq \frac{v}{2}} \frac{(2\pi i m)^{\mu} (k+v-2\mu-1)!}{(k+v-\mu-1)! \mu!} \xi_{v-2\mu}^{(\mu)}(\tau),$$

and then the transformation equations (6) of the X_v follow from $\xi_v \in M_{k+v}(\Gamma)$. In particular, taking $\xi_0 = f$, $\xi_v = 0$ ($v > 1$) and $m = 1$ we obtain the following result, due (independently of one another) to Kuznetsov and Cohen:

THEOREM 3.3 (Kuznetsov [16], Cohen [7]). *Let $f(\tau)$ be a modular form of weight k on Γ . Then the function*

$$(13) \quad \tilde{f}(\tau, z) := \sum_{v=0}^{\infty} \frac{(2\pi i)^v (k-1)!}{v! (k+v-1)!} f^{(v)}(\tau) z^{2v}$$

satisfies the transformation equation

$$(14) \quad \tilde{f}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e\left(\frac{cz^2}{c\tau+d}\right) \tilde{f}(\tau, z), \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right)$$

We mention a corollary which will be used later.

COROLLARY (Cohen [6, Th. 7.1]). *Let f_1, f_2 be modular forms on Γ of weight k_1 and k_2 , respectively, $v \in \mathbb{N}_0$. Then the function*

$$F_v(f_1, f_2) := (2\pi i)^{-v} \sum_{\mu=0}^v (-1)^{v-\mu} \binom{v}{\mu} \frac{\Gamma(k_1+v)}{\Gamma(k_1+\mu)} \frac{\Gamma(k_2+v)}{\Gamma(k_2+\mu-v)} f_1^{(\mu)} f_2^{(v-\mu)}$$

is a modular form of weight k_1+k_2+2v on Γ and is a cusp form if $v > 0$.

(We have modified Cohen's definition by a factor $(2\pi i)^{-v}$ to make the Fourier coefficients of $F_v(f_1, f_2)$ rational in those of f_1 and f_2 .) The corollary follows by computing the coefficient of z^{2v} in $\tilde{f}_1(\tau, z)\tilde{f}_2(\tau, iz)$, which by Theorem 3.3 transforms like a modular form of weight k_1+k_2 under Γ .

We observe that the known result 3.3 could also have been used to prove 3.1 and 3.2. (We preferred to give a direct proof in the context of the theory of Jacobi forms, especially as the use of the differential operators L_k makes the proof rather natural.) Indeed, let $M_{k,m}^{(v)}$ be the subspace of $M_{k,m}$ of functions ϕ which are $O(z^v)$, i.e. have a development $\chi_v(\tau)z^v + \chi_{v+1}(\tau)z^{v+1} + \dots$. From (6) it is clear that the leading coefficient χ_v is then a modular form of weight

$k+v$ and we get an exact sequence

$$0 \rightarrow M_{k,m}^{(v+1)} \rightarrow M_{k,m}^{(v)} \rightarrow M_{k+v}$$

in which the first arrow is the inclusion and the second is $\phi \mapsto X_v$.

On the other hand, $M_{k,m}^{(v)} \simeq M_{k+v,m}$ by division by z^v (this was already used for $v=1$ when we reduced the study of $M_{k,m}^+$ to that of $M_{k,m}^+$), and 3.3 gives a map $M_{k+v} \rightarrow M_{k+v,m}$ by $f \mapsto \tilde{f}(\tau, \sqrt{m}z)$; this shows that the last map above is surjective and gives an explicit splitting.

To get the sequence of modular forms ξ_0, ξ_1, \dots associated to $\phi \in M_{k,m}$ we now proceed by induction: having found $\xi_0, \xi_1, \dots, \xi_{v-1}$ such that

$$\phi(\tau, z) - \sum_{v' < v} \tilde{\xi}_{v'}(\tau, \sqrt{m}z) z^{v'} \equiv 0 \pmod{z^v}$$

we define $\xi_v(\tau)$ as the leading coefficient (coefficient of z^v) in the expression on the left-hand side; then $\phi = \sum_v \tilde{\xi}_v(\tau, \sqrt{m}z) z^v$ as a formal power series and this is equivalent to the series of identities (12) or (7).

We have gone into the meaning of the development coefficients $D_v \phi$ fairly deeply because they play an important role in the study of Jacobi forms and because the relation with the identity (14) of Kuznetsov and Cohen concerning \tilde{f} (which is *not* a Jacobi form) seemed striking. In particular, we should mention that (13) can be written

$$\tilde{f}(\tau, z) = a(0) + (k-1)! \sum_{n=1}^{\infty} a(n) \frac{J_{k-1}(4\pi\sqrt{n}z)}{(2\pi\sqrt{n}z)^{k-1}} q^n$$

if $f = \sum a(n)q^n$ (this is the form in which Kuznetsov gave the identity)

To see where the Bessel functions come from, note that the function

$h(z) = (k-1)! \frac{J_{k-1}(4\pi z)}{(2\pi z)^{k-1}}$ satisfies the ordinary differential equation $h'' + \frac{2k-1}{z} h' + (4\pi)^2 h = 0$ and is the only solution holomorphic at the

origin and with $h(0) = 1$. By separation of variables we see that

$\tilde{f}(\tau, z) = \sum_0^{\infty} a(n) h(\sqrt{n}z) e^{2\pi i n \tau}$ is the unique solution of the partial differential equation $L_k \tilde{f} = 0$ satisfying the boundary conditions

$\tilde{f}(\tau+1, z) = \tilde{f}(\tau, z)$ and $\tilde{f}(\tau, 0) = f(\tau)$, and this uniqueness together with the fact that L_k commutes with the operation of $SL_2(\mathbb{R})$ (eq. (11)) immediately implies that \tilde{f} has the property (14).

As a first application of the maps \mathcal{D}_v to Jacobi forms, we have a second proof and sharpening of Theorem 1.1:

THEOREM 3.4. $\dim J_{k,m}(\Gamma) \leq \dim M_k(\Gamma) + \sum_{v=1}^{2m} \dim S_{k+v}(\Gamma)$.

Indeed, $\xi_0 = \dots = \xi_{2m} = 0$ implies $X_0 = \dots = X_{2m} = 0$ or $\phi = O(z^{2m+1})$, so Theorem 1.2 implies that the map

$$\mathcal{D} = \bigoplus_{v=0}^{2m} \mathcal{D}_v: J_{k,m}(\Gamma) \rightarrow M_k(\Gamma) \oplus S_{k+1}(\Gamma) \oplus \dots \oplus S_{k+2m}(\Gamma)$$

is injective. Note that half of the spaces $M_{k+v}(\Gamma)$ are 0 if $-I_2 \in \Gamma$; in particular, for $\Gamma = \Gamma_1$ we have

$$(15) \quad \dim J_{k,m} \leq \begin{cases} \dim M_k + \dim S_{k+2} + \dots + \dim S_{k+2m} & (k \text{ even}), \\ \dim S_{k+1} + \dim S_{k+3} + \dots + \dim S_{k+2m-1} & (k \text{ odd}) . \end{cases}$$

Here the second estimate can even be strengthened to

$$(16) \quad \dim J_{k,m} \leq \dim S_{k+1} + \dots + \dim S_{k+2m-3} ,$$

because an odd Jacobi form must vanish at the three 2-division points $\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ and hence cannot have more than a $(2m-3)$ -fold zero at $z=0$.

Application: Jacobi Forms of Index One

Theorem 3.4 is the basis for the analysis of the structure of $J_{*,*} = \bigoplus_{k,m} J_{k,m}$ as given in Chapter III, to which the reader may now skip if he so desires (the results of §§4-7 are not used there). As an

example, we now treat the case $m=1$, which is particularly easy and will be used in Chapter II. Equations (15) and (16) (or Theorem 2.2) give

$$J_{k,1} = 0 \quad (k \text{ odd}), \quad \dim J_{k,1} \leq \dim M_k + \dim S_{k+2} \quad (k \text{ even}).$$

On the other hand, the Fourier developments of $E_{4,1}$ and $E_{6,1}$ as given after Theorem 2.1, show that the quotient

$$\frac{E_{6,1}(\tau, z)}{E_{4,1}(\tau, z)} = 1 - (144\zeta + 456 + 144\zeta^{-1})q + \dots$$

depends on z and hence is not a quotient of two modular forms, so the map

$$M_{k-4} \oplus M_{k-6} \rightarrow J_{k,1}$$

$$(f, g) \mapsto f(\tau)E_{4,1}(\tau, z) + g(\tau)E_{6,1}(\tau, z)$$

is injective. Since $\dim M_{k-4} + \dim M_{k-6} = \dim M_k + \dim S_{k+2}$ for all k (this follows from the well-known formula for $\dim M_k$), we deduce

THEOREM 3.5. *The space of Jacobi forms of index 1 on $SL_2(\mathbb{Z})$ is a free module of rank 2 over M_* , with generators $E_{4,1}$ and $E_{6,1}$. The map*

$$\mathcal{D}_0 + \mathcal{D}_2 : J_{k,1} \rightarrow M_k + S_{k+2}$$

($\mathcal{D}_0, \mathcal{D}_2$ as in Theorem 3.1) is an isomorphism.

In particular, we find that the space $J_{8,1}$ is one-dimensional, with generator $E_{8,1} = E_4 \cdot E_{4,1}$ ($E_4 = 1 + 240q + \dots$ the Eisenstein series in M_4), while the first cusp forms of index 1 are the forms

$$\phi_{10,1} = \frac{1}{144} (E_6 E_{4,1} - E_4 E_{6,1}),$$

(17)

$$\phi_{12,1} = \frac{1}{144} (E_4^2 E_{4,1} - E_6 E_{6,1})$$

of weight 10 and 12, respectively (the factor 144 has been inserted to make the coefficients of $\phi_{10,1}$ and $\phi_{12,1}$ integral and coprime). We

have tabulated the first coefficients $e_{k,1}$ of $E_{k,1}$ ($k=4,6,8$) and $c_{k,1}$ of $\phi_{k,1}$ ($k=10,12$) in Table 1; notice that it suffices to give a single sequence of coefficients $c(N)$ ($N \geq 0$, $N \equiv 0,3(\text{mod } 4)$) since by 2.2 any Jacobi form of index 1 has Fourier coefficients of the form $c(n,r) = c(4n - r^2)$ for some $\{c(N)\}$. To compute the $c(N)$, we can use either assertion of Theorem 3.5, e.g. for $\phi_{10,1}$, $\phi_{12,1}$ we can either use (17) and the known Fourier expansions of E_k and $E_{k,m}$ or else (what is quicker) use the expansions

$$(18) \quad \begin{aligned} D_0 \phi_{10,1} &= 0, & D_2 \phi_{10,1} &= 20\Delta, \\ D_0 \phi_{12,1} &= 12\Delta, & D_2 \phi_{12,1} &= 0 \end{aligned}$$

$(\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n)$ to obtain the identities

$$\sum_{|r| < 2\sqrt{n}} c_{10,1}(4n - r^2) = 0, \quad \sum_{0 < r < 2\sqrt{n}} r^2 c_{10,1}(4n - r^2) = \tau(n),$$

$$\sum_{|r| < 2\sqrt{n}} c_{12,1}(4n - r^2) = 12\tau(n), \quad \sum_{0 < r < 2\sqrt{n}} r^2 c_{12,1}(4n - r^2) = n\tau(n)$$

and then solve these recursively for $c_{k,1}(N)$.

The functions

$$\phi_{10,1} = (\zeta - 2 + \zeta^{-1})q + (-2\zeta^2 - 16\zeta + 36 - 16\zeta^{-1} - 2\zeta^{-2})q^2 + \dots$$

$$\phi_{12,1} = (\zeta + 10 + \zeta^{-1})q + (10\zeta^2 - 88\zeta - 132 - 88\zeta^{-1} + 10\zeta^{-2})q^2 + \dots$$

have several beautiful properties and will play a role in the structure theory developed in Chapter III. Here we mention only the following:

THEOREM 3.6. *The quotient*

$$\frac{\phi_{12,1}(\tau, z)}{\phi_{10,1}(\tau, z)} = \frac{\zeta + 10 + \zeta^{-1}}{\zeta - 2 + \zeta^{-1}} + 12(\zeta - 2 + \zeta^{-1})q + \dots$$

is $-3/\pi^2$ times the Weierstrass p -function $p(\tau, z)$.

Indeed, since $\phi_{10,1}$ vanishes doubly at $z=0$ and (by Theorem 1.2) nowhere else in $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$, and since by (18)

$$(19) \quad \begin{aligned} \phi_{10,1} &= (2\pi i)^2 \Delta(\tau) z^2 + O(z^4) , \\ \phi_{12,1} &= 12\Delta(\tau) + O(z^2) , \end{aligned}$$

the quotient in question is a doubly periodic function of z with a double pole with principal part $\frac{12}{(2\pi i)^2} z^{-2}$ at $z=0$ and no other poles in a period parallelogram, so must equal $\frac{12}{(2\pi i)^2} p(\tau, z)$.

Finally, we note that, just as the two Eisenstein series $E_{4,1}$ and $E_{6,1}$ form a free basis of $J_{*,1}$ over M_* , the two cusp forms $\phi_{10,1}$ and $\phi_{12,1}$ form a basis of $J_{*,1}^{\text{cusp}}$ over M_* , i.e. we have an isomorphism

$$M_{k-10} \oplus M_{k-12} \xrightarrow{\sim} J_{*,1}^{\text{cusp}}$$

$$(f, g) \mapsto f(\tau)\phi_{10,1}(\tau, z) + g(\tau)\phi_{12,1}(\tau, z) .$$

Thus the Jacobi forms

$$E_4(\tau)^a E_6(\tau)^b \phi_{j,1}(\tau, z) \quad (a, b \geq 0, \quad j \in \{10, 12\}, \quad 4a + 6b + j = k)$$

form an additive basis of the space of Jacobi cusp forms of weight k and index 1. Each of them has a Fourier expansion of the form $\sum c(4n - r^2) q^n \zeta^r$; the coefficients $c(N)$ for $N \leq 20$ and all weights $k \leq 50$ are given in Table 2.

CHAPTER II
RELATIONS WITH OTHER TYPES OF MODULAR FORMS

§5. Jacobi Forms and Modular Forms of Half-Integral Weight

In §2 we showed that the coefficients $c(n, r)$ of a Jacobi form of index m depend only on the "discriminant" $r^2 - 4nm$ and on the value of $r \pmod{2m}$, i.e.

$$(1) \quad c(n, r) = c_r(4nm - r^2), \quad c_{r'}(N) = c_r(N) \quad \text{for } r' \equiv r \pmod{2m}.$$

From this it follows very easily, as we will now see, that the space of Jacobi forms of weight k and index m is isomorphic to a certain space of (vector-valued) modular forms of weight $k - \frac{1}{2}$ in one variable; the rest of this section will then be devoted to identifying this space with more familiar spaces of modular forms of half-integral weight and studying the correspondence more closely.

Equation (1) gives us coefficients $c_\mu(N)$ for all $\mu \in \mathbb{Z}/2m\mathbb{Z}$ and all integers $N \geq 0$ satisfying $N \equiv -\mu^2 \pmod{4m}$ (notice that μ^2 is well-defined modulo $4m$ if μ is given modulo $2m$), namely

$$(2) \quad c_\mu(N) := c\left(\frac{N+r^2}{4m}, r\right) \quad (\text{any } r \in \mathbb{Z}, r \equiv \mu \pmod{2m})$$

(since μ is a residue class, one should more properly write $r \in \mu$ rather than $r \equiv \mu \pmod{2m}$; we permit ourselves the slight abuse of notation). We extend the definition to all N by setting $c_\mu(N) = 0$ if $N \not\equiv -\mu^2 \pmod{4m}$, and set

$$(3) \quad h_\mu(\tau) := \sum_{N=0}^{\infty} c_\mu(N) q^{N/4m} \quad (\mu \in \mathbb{Z}/2m\mathbb{Z})$$

and

$$(4) \quad \theta_{m,\mu}(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu \pmod{2m}}} q^{r^2/4m} \zeta^r .$$

(The $\theta_{m,\mu}$ are independent of the function ϕ .) Then

$$\begin{aligned} \phi(\tau, z) &= \sum_{\mu \pmod{2m}} \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu \pmod{2m}}} \sum_{n \geq r^2/4m} c_\mu(4nm - r^2) q^n \zeta^r \\ (5) \quad &= \sum_{\mu \pmod{2m}} \sum_{r \equiv \mu \pmod{2m}} \sum_{N \geq 0} c_\mu(N) q^{\frac{N+r^2}{4m}} \zeta^r \\ &= \sum_{\mu \pmod{2m}} h_\mu(\tau) \theta_{m,\mu}(\tau, z) . \end{aligned}$$

Thus knowing the $(2m)$ -tuple $(h_\mu)_{\mu \pmod{2m}}$ of functions of one variable is equivalent to knowing ϕ . Reversing the above calculation, we see that given any functions h_μ as in (3) with $c_\mu(N) = 0$ for $N \not\equiv -\mu^2 \pmod{4m}$ equation (5) defines a function ϕ (with Fourier coefficients as in (1)) which transforms like a Jacobi form with respect to $z \mapsto z + \lambda\tau + \mu$ ($\lambda, \mu \in \mathbb{Z}$) and satisfies the right conditions at infinity. In order for ϕ to be a Jacobi form, we still need a transformation law with respect to $SL_2(\mathbb{Z})$. Since the theta-series (4) have weight $\frac{1}{2}$ and index m , while ϕ has weight k and index m , we see from (5) that the h_μ must be modular forms of weight $k - \frac{1}{2}$. To specify their precise transformation law, it suffices to consider the generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of Γ_1 . For the first we have

$$(6) \quad \theta_{m,\mu}(\tau+1, z) = e_{4m}(\mu^2) \theta_{m,\mu}(\tau, z)$$

and

$$(7) \quad h_\mu(\tau+1) = e_{4m}(-\mu^2) h_\mu(\tau) ,$$

as one sees either from the invariance of the sum (5) under $\tau \mapsto \tau + 1$ or from the congruence $N \equiv -\mu^2 \pmod{4m}$ in (3). For the second we have as an easy consequence of the Poisson summation formula the identity

$$(8) \quad \theta_{m,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \sqrt{\tau/2m} e^{2\pi i mz^2/\tau} \sum_{v \pmod{2m}} e_{2m}(-\mu v) \theta_{m,v}(\tau, z),$$

so (5) and the transformation law of ϕ under $(\tau, z) \mapsto \left(-\frac{1}{\tau}, \frac{z}{\tau}\right)$ give

$$(9) \quad h_\mu\left(-\frac{1}{\tau}\right) = \frac{\tau^k}{\sqrt{2m\tau/1}} \sum_{v \pmod{2m}} e_{2m}(\mu v) h_v(\tau).$$

We have proved

THEOREM 5.1. *Equation (5) gives an isomorphism between $J_{k,m}$ and the space of vector valued modular forms $(h_\mu)_{\mu \pmod{2m}}$ on $SL_2(\mathbb{Z})$ satisfying the transformation laws (7) and (9) and bounded as $\text{Im}(\tau) \rightarrow \infty$.*

When we speak of "vector-valued" forms in Theorem 5.1, we mean that the vector $\vec{h}(\tau) = (h_\mu)_{\mu \pmod{2m}}$ satisfies

$$(10) \quad \vec{h}(M\tau) = (c\tau+d)^{k-\frac{1}{2}} U(M) \vec{h}(\tau) \quad (M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1)$$

where $U(M) = (U_{\mu\nu}(M))$ is a certain $2m \times 2m$ matrix (the map $U: \Gamma_1 \rightarrow GL_{2m}(\mathbb{C})$ is not quite a homomorphism because of the ambiguities arising from the choice of square-root in (10); to get a homomorphism one must replace Γ_1 by a double cover). The result 5.1 would be more pleasing if we could identify $J_{k,m}$ with a space of ordinary (i.e. scalar) modular forms of weight $k - \frac{1}{2}$ on some congruence subgroup of Γ_1 . We will do this below in the cases $m=1$ and m prime, k even, and also discuss the general case a little. First, however, we look at some immediate consequences of Theorem 5.1.

First of all, by combining (5) with the equations $\theta_{m,-\mu}(\tau, z) = \theta_{m,\mu}(\tau, -z)$ and $\phi(\tau, -z) = (-1)^k \phi(\tau, z)$ we deduce the symmetry property

$$(11) \quad h_{-\mu} = (-1)^k h_\mu \quad (\mu \in \mathbb{Z}/2m\mathbb{Z})$$

(this can also be proved by applying (9) twice), so that in fact (h_μ) reduces to an $(m+1)$ -tuple of forms $(h_\mu + h_{-\mu})_{0 \leq \mu \leq m}$ if k is even and to an $(m-1)$ -tuple $(h_\mu - h_{-\mu})_{0 < \mu < m}$ if k is odd. However, we can introduce a finer splitting if m is composite. For each divisor m' of m with $(m', m/m') = 1$ (there are 2^t such divisors, where t is the number of distinct prime factors of m) choose an integer $\xi = \xi_m$, satisfying

$$(12) \quad \xi \equiv 1 \pmod{2m/m'} , \quad \xi \equiv -1 \pmod{2m'} ;$$

such a ξ clearly exists and is unique $(\pmod{2m})$, and the set of ξ_m , for all $m' \mid m$ is precisely $\{\xi \pmod{2m} \mid \xi^2 \equiv 1 \pmod{4m}\}$. Now map the collection of $(2m)$ -tuples $(h_\mu)_\mu$ into itself by the permutation

$$(13) \quad (h_\mu)_{\mu \pmod{2m}} \mapsto (h_{\xi\mu})_{\mu \pmod{2m}} .$$

Because $\xi^2 \equiv 1 \pmod{4m}$, it is clear that equations (7) and (9) are preserved. Hence we deduce

THEOREM 5.2. *For each divisor m' of m with $(m', m/m') = 1$ there is an operator W_m , from $J_{k,m}$ to itself such that the coefficient of $q^{n,r} \zeta^r$ in $\phi|W_m$, is $c(n', r')$ where $r' \equiv -r \pmod{2m'}$, $r' \equiv r \pmod{2m/m'}$, $4n'm - r'^2 = 4nm - r^2$. These operators are all involutions and together form a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^t$ and generated by the $W_{p_i^{v_i}}$ ($m = \prod_{i=1}^t p_i^{v_i}$).*

Next, we relate the expansion (5) to the Petersson product introduced in §2.

THEOREM 5.3. Let

$$\phi = \sum_{\mu} h_{\mu} \theta_{m,\mu}, \quad \psi = \sum_{\mu} g_{\mu} \theta_{m,\mu}$$

be two Jacobi forms in $J_{k,m}$. Then

$$(\phi, \psi) = \frac{1}{\sqrt{2m}} \int_{\Gamma_1 \backslash \mathcal{H}} \sum_{\mu \pmod{2m}} h_{\mu}(\tau) \overline{g_{\mu}(\tau)} v^{k-\frac{5}{2}} du dv.$$

In other words, the Petersson scalar product of ϕ and ψ as defined in §2 is equal (up to a constant) to the Petersson product in the usual sense of the vector-valued modular forms $(h_{\mu})_{\mu}$, $(g_{\mu})_{\mu}$ of weight $k-\frac{5}{2}$. The assertions of Theorem 2.5 (that (ϕ, ψ) is well defined and infinite if ϕ or ψ is cuspidal) now follow from the corresponding statements for modular forms in one variable.

Proof. We first compute the scalar product of $\theta_{m,\mu}$ and $\theta_{m,\nu}$ at a fixed fiber ($\tau \in \mathcal{H}$ fixed):

$$\begin{aligned} & \int_{\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}} \theta_{m,\mu}(\tau, z) \overline{\theta_{m,\nu}(\tau, z)} e^{-4\pi my^2/v} dx dy \\ &= \int_{\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}} \sum_{\substack{r \equiv \mu \pmod{2m} \\ s \equiv \nu \pmod{2m}}} e(rz - s\bar{z}) e\left(\frac{r^2\tau - s^2\bar{\tau}}{4m}\right) e^{-4\pi my^2/v} dx dy \end{aligned}$$

Using

$$\int_{\mathbb{R}/\mathbb{Z}} e(rz - s\bar{z}) dx = \delta_{rs} e^{-4\pi ry}$$

we find that this equals

$$\delta_{\mu\nu} \int_{\mathbb{R}/v\mathbb{Z}} \sum_{r \equiv \mu \pmod{2m}} e^{-\frac{4\pi rm}{v}} \left(y + \frac{rv}{2m}\right)^2 dy$$

$$= \delta_{\mu\nu} \int_{-\infty}^{\infty} e^{-4\pi my^2/v} dy = \sqrt{v/4m} \delta_{\mu\nu}$$

(here δ_{rs} is the Kronecker delta of r,s and $\delta_{\mu\nu}$ of μ and ν modulo 2). It immediately follows that

$$(\phi, \psi) = \frac{1}{\sqrt{4m}} \int_{\Gamma_1 \backslash \mathcal{H}} \sum_{\mu \pmod{2m}} h_\mu(\tau) \overline{g_\mu(\tau)} v^{k-\frac{1}{2}} \frac{du dv}{v^2}$$

as claimed.

Since W_m simply permutes the h_μ , it follows from Theorem 5.3 that W_m is Hermitian. From Theorem 4.5 it is clear that the W_m commute with all T_ℓ ($(\ell, m) = 1$). Hence we deduce

COROLLARY. $J_{k,m}$ has a basis of simultaneous eigenforms for all T_ℓ ($(\ell, m) = 1$) and W_m , ($m' \parallel m$).

Theorem 5.2 gives a splitting of $J_{k,m}$ as $\bigoplus_{\pm \dots \pm} J_{k,m}^{\pm \dots \pm}$, where the sum is over all t -tuples of signs with product $(-1)^k$; Theorem 5.3 shows that this splitting is orthogonal and that each summand has a basis consisting of Hecke eigenforms.

We now discuss the connection between Jacobi forms and scalar-valued modular forms of weight $k-\frac{1}{2}$. We recall that modular forms of half-integral weight are defined like forms of integral weight, except that the automorphy factor describing the action of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ involves the Legendre symbol $(\frac{c}{d})$; the easiest way to specify the automorphy factor exactly is to say that for a modular form $h(\tau)$ on $\Gamma_0(4m)$ the quotient h/θ^{2k-1} , where $\theta(\tau) = \sum q^n$, is invariant under $\Gamma_0(4m)$. We denote the space of such forms by $M_{k-\frac{1}{2}}(\Gamma_0(4m))$. Shimura developed an extensive theory of such forms in [29], [30]. In particular he showed that one can define Hecke operators T_p on $M_{k-\frac{1}{2}}(\Gamma_0(4m))$ for all primes $p \nmid 4m$, that $M_{k-\frac{1}{2}}(\Gamma_0(4m))$ is spanned by simultaneous eigenforms of these operators, and that the set of eigenvalues of an eigenform is the same as the set of eigenvalues of a certain Hecke eigenform of

weight $2k-2$. (Shimura used the notation $T(p^2)$ for the Hecke operators in half-integral weight because they are defined using matrices of determinant p^2 , but we prefer to write T_p since these are the only naturally definable operators and correspond to the operators T_p in weight $2k-2$.) His conjecture that the eigenforms of integral weight obtained in this way have level $2m$ was proved by Niwa [24]. For the case $m=1$ (and later for the case of odd, square-free m [12]), Kohnen [11] showed how one could get all the way down to level m by passing to the subspace

$$M_{k-\frac{1}{2}}^+(4m) = \left\{ h \in M_{k-\frac{1}{2}}(\Gamma_0(4m)) \mid h = \sum_{\substack{N=0 \\ (-1)^{k-1} N \equiv 0, 1 \pmod{4}}}^{\infty} c(N) q^N \right\}$$

of forms in $M_{k-\frac{1}{2}}(\Gamma_0(4m))$ whose N^{th} Fourier coefficient vanishes for all N with $(-1)^{k-1}N$ congruent to 2 or 3 $\pmod{4}$. Following Kohnen's notation in [12], we shall write simply $M_{k-\frac{1}{2}}(m)$ for $M_{k-\frac{1}{2}}^+(4m)$ and $M_{k-\frac{1}{2}}$ instead of $M_{k-\frac{1}{2}}(1)$. Then Kohnen's main result for $m=1$ says that one can define commuting and hermitian Hecke operators T_p on $M_{k-\frac{1}{2}}$ for all p (agreeing with Shimura's operators if $p \neq 2$) and that $M_{k-\frac{1}{2}}$ then becomes isomorphic to M_{2k-2} as a module over the ring of Hecke operators i.e. there is a 1-1 correspondence between eigenforms $h \in M_{k-\frac{1}{2}}$ and $\tilde{h} \in M_{2k-2}$ such that the eigenvalues of h and \tilde{h} under T_p agree for all p . Explicitly, $T_p: M_{k-\frac{1}{2}} \rightarrow M_{k-\frac{1}{2}}$ (k even) is given by

$$(14) \quad T_p: \sum_{\substack{N \geq 0 \\ N \equiv 0 \text{ or } 3 \pmod{4}}} c(N) q^N \mapsto \sum_{\substack{N \geq 0 \\ N \equiv 0 \text{ or } 3 \pmod{4}}} \left(c(Np^2) + \left(\frac{-N}{p} \right) p^{k-2} c(N) + p^{2k-3} c\left(\frac{N}{p^2}\right) \right) q^N$$

Observe also that $M_{k-\frac{1}{2}} := \bigoplus_k M_{k-\frac{1}{2}}$ is a module over M_∞ by $h(\tau) \mapsto f(4\tau)h(\tau)$ ($h \in M_{k-\frac{1}{2}}$, $f \in M_\infty$).

We can now state:

CHAPTER III
THE RING OF JACOBI FORMS

§8. Basic Structure Theorems

The object of this and the following section is to obtain as much information as possible about the algebraic structure of the set of Jacobi forms, in particular about

- i) the dimension of $J_{k,m}$ (k, m fixed), i.e. the structure of this space as a vector space over \mathbb{C} ;
- ii) the additive structure of $J_{*,m} = \bigoplus_k J_{k,m}$ (m fixed) as a module over the graded ring $M_* = \bigoplus_k M_k$ of ordinary modular forms;
- iii) the multiplicative structure of the bigraded ring

$$J_{*,*} = \bigoplus_{k,m} J_{k,m} \text{ of all Jacobi forms.}$$

We will study only the case of forms on the full Jacobi group Γ_1^J (and usually only the case of forms of even weight), but many of the considerations could be extended to arbitrary Γ .

The simplest properties of the space of Jacobi forms were already given in Chapter I. There we showed that $J_{k,m}$ is finite-dimensional for all k and m and zero if k or m is negative (Theorem 1.1 and its proof) and obtained the explicit dimension estimate

$$(1) \quad \dim J_{k,m} \leq \begin{cases} \dim M_k + \sum_{v=1}^m \dim S_{k+2v} & (k \text{ even}) \\ \sum_{v=1}^{m-1} \dim S_{k+2v-1} & (k \text{ odd}) \end{cases},$$

(Theorem 3.4 and the following remarks). We also proved that $J_{*,m}$ reduces to M_* if $m=0$ and is free over M_* on two generators $E_{4,1}$, $E_{6,1}$ for $m=1$. The very precise result for $J_{*,1}$ was obtained by comparing the upper bound (1) with the lower bound coming from the linear independence of the two special modular forms $E_{4,1}$ and $E_{6,1}$. Similarly, we will get information for higher m by combining (1) with the following result.

THEOREM 8.1. *The forms $E_{4,1}$ and $E_{6,1}$ are algebraically independent over M_* .*

Proof. Clearly the theorem is equivalent to the algebraic independence over M_* of the two cusp forms $\phi_{10,1}$ and $\phi_{12,1}$ defined in §3, (17). Suppose that these forms are dependent. Since both have index 1 and any relation can be assumed to be homogeneous, the relation between them has the form

$$(2) \quad \sum_{j=0}^m g_j(\tau) \phi_{10,1}(\tau, z)^j \phi_{12,1}(\tau, z)^{m-j} = 0$$

for some m , where the g_j are modular forms, not all zero. Let j_0 be the smallest j for which g_j is not identically zero. Substituting into (2) the Taylor expansions given in §3, (19), we find that the left-hand side of (2) equals

$$(\text{constant})\Delta(\tau)^m g_{j_0}(\tau) z^{2j_0} + O(z^{2j_0+2})$$

and hence cannot vanish identically. This proves the theorem.

Since the two functions $E_{4,1}$ and $E_{6,1}$ will play a basic role in our analysis of $J_{*,*}$, we introduce the abbreviations A,B to denote them. Thus $A \in J_{4,1}$, $B \in J_{6,1}$ and the theorem just proved says that the map $M_*[X,Y] \rightarrow J_{*,*}$ sending X to A and Y to B is injective. The (k,m) -

graded component of this statement is that the map

$$M_{k-4m} \times M_{k-4m-2} \times \cdots \times M_{k-6m} \longrightarrow J_{k,m}$$

$$(f_0, f_1, \dots, f_m) \longmapsto f_0 A^m + f_1 A^{m-1} B + \cdots + f_m B^m$$

is injective. This implies

$$\text{COROLLARY 1. } \dim J_{k,m} \geq \sum_{j=0}^m \dim M_{k-4m-2j}.$$

We now show how this estimate can be combined with (1) to obtain algebraic information about the ring of Jacobi forms.

COROLLARY 2. Fix an integer $m \geq 0$. Then the space $J_{2*,m}$ of Jacobi forms of index m and even weight is a module of rank $m+1$ over M_* .

Proof. The linear independence of the monomials $A^j B^{m-j}$ ($0 \leq j \leq m$) over M_* implies that the rank is at least $m+1$. Using the two facts $\dim M_k \rightarrow \infty$ and $\dim M_{k+0}(1) = \dim M_k + O(1)$ (we do not need the more precise formula $\dim M_k = \frac{k}{12} + O(1)$), we can write Corollary 1 in the weakened form

$$\dim J_{k,m} \geq (m+1) \dim M_k + O(1) \quad (k \rightarrow \infty, k \text{ even}).$$

If there were $m+2$ Jacobi forms of index m linearly independent over M_* then the same argument used to prove Corollary 1 would show that the factor $m+1$ in this inequality could be replaced by $m+2$, contradicting the upper bound

$$\dim J_{k,m} \leq (m+1) \dim M_k + O(1)$$

coming from (1). Hence the rank is exactly $m+1$.

COROLLARY 3. Every Jacobi form can be expressed uniquely as a polynomial in A and B with coefficients which are meromorphic modular forms (quotients of holomorphic modular forms).

Proof. If $\phi \in J_{*,m}$, then the forms $\phi, A^m, A^{m-1}B, \dots, B^m$ must be linearly dependent over M_* by Corollary 2, and this linear relation must involve ϕ by the theorem, i.e. we have the formula

$$(3) \quad f(\tau)\phi(\tau, z) = \sum_{j=0}^m f_j(\tau) E_{4,1}(\tau, z)^{m-j} E_{6,1}(\tau, z)^j$$

with $f, f_j \in M_*$ and $f \neq 0$; dividing by f gives the assertion of the corollary (the uniqueness follows at once from the algebraic independence of A and B).

Looking more carefully at the proof just given, we can obtain an estimate of the minimal weight of the form f in (3). Indeed, the proof of Corollary 2 depended on the fact that the upper and lower bounds we had obtained for $\dim J_{k,m}$ (m fixed) differ by a bounded amount, i.e.

$$\left(\dim M_k + \sum_{v=1}^m \dim S_{k+2v} \right) - \left(\sum_{v=0}^m \dim M_{k-4m-2v} \right) \leq C$$

for some constant C depending only on m ; using the explicit formulas for $\dim M_k$ we see that this holds with $C = \frac{m(m-1)}{2}$ and with equality for all even k . Hence the codimension of $M_*[A, B] \cap J_{k,m}$ in $J_{k,m}$ is bounded by C for all k . Now if $\phi \in J_{k,m}$ and there is no relation of the form (3) with f of weight h , then the subspaces $\phi \cdot M_h$ and $M_*[A, B] \cap J_{k+h,m}$ of $J_{k+h,m}$ are disjoint and hence the dimension of M_h is $\leq C$. Therefore there is a relation of type (3) at latest in weight $h = 12C = 6m(m-1)$. (Later we shall obtain a much better bound.)

Corollary 3 says that $J_{*,*} \otimes K_*$, where $K_* = \mathbb{C}(E_4, E_6)$ is the

quotient field of M_* , is a free polynomial algebra $K_*[A, B]$ over K_* . In particular, the quotient field of $J_{*,*}$ is $\mathbb{C}(E_4, E_6, A, B)$. In view of Theorem 3.6, this is equivalent to the statement that the field of Jacobi functions (= meromorphic Jacobi forms of weight 0 and index 0) for $SL_2(\mathbb{Z})$ is $\mathbb{C}(j(\tau), \rho(\tau, z))$, a fact which is more or less obvious from the definition of Jacobi functions and the fact that every even elliptic function on $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ is a rational function of $\rho(\tau, z)$.

Before proceeding with the theory we would like to discuss the case of forms of index 2 in some detail; this will both motivate and illustrate our results. Here, of course, we do not need Theorem 8.1, since we can check the linear independence of A^2 , AB and B^2 (or of the monomials A^m, \dots, B^m for any fixed m) directly by looking at the first few terms of their Fourier expansions, as was done in the case $m=1$ in §3. Thus we obtain the lower bound of Corollary 1 "by hand". This bound and the upper bound are given for small k by the table

k	2	4	6	8	10	12
$\dim M_k + \dim S_{k+2} + \dim S_{k+4}$	0	1	1	2	2	3
$\dim M_{k-8} + \dim M_{k-10} + \dim M_{k-12}$	0	0	0	1	1	2

Thus the upper and lower bounds no longer agree, as they did for $m=1$, but now they always differ by 1. We will see that the upper bound is in fact always the correct one. In §2 we showed that $J_{k,m} \neq 0$ for all even $k \geq 4$ and all $m \geq 1$. Hence there exist non-zero forms $X \in J_{4,2}$, $Y \in J_{6,2}$. By Corollary 2, there must be two linear relations over M_* among the five Jacobi forms X , Y , A^2 , AB and B^2 of index 2. To find them, we could calculate the leading Fourier coefficients of X and Y (we didn't give complete formulas for the coefficients of Eisenstein series of index > 1 in §2, but from §4 we know that $E_{4,2}$ and $E_{6,2}$ are